

The prime structure of $\gcd(d_n a_n, d_n b_n)$ for Apéry's sequences: a symmetric digit criterion and partial results toward Problem 3.2

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Abstract

Let $(a_n), (b_n)$ be Apéry's $\zeta(3)$ sequences and $d_n = \text{lcm}(1, \dots, n)^3$. We determine, for every prime $p \in (\sqrt{n}, n]$, exactly when $v_p(\gcd(d_n a_n, d_n b_n)) > 0$: writing $n = jp + n_0$ in base p , this happens if and only if $p \mid b_{n_0}$ or $p \mid \text{num}(a_j)$ — a criterion symmetric in the two solutions of Apéry's recurrence, in which the integer solution b sees the low digit and the rational solution a sees the high digit. Small primes $p \leq \sqrt{n}$ contribute $O(\sqrt{n} \log n)$ unconditionally. As corollaries: (i) the full Conjecture 3.2 is equivalent to a sparsity statement about prime divisors of the two Apéry solutions; (ii) the analogue of 3.2 restricted to the a -side criterion holds for almost all n unconditionally; (iii) the b -side reduces to any nontrivial bound on $Z(p) := \#\{0 \leq r < p : p \mid b_r\}$, a quantity whose study was initiated by Malik–Straub and Rowland–Yassawi and which we verify satisfies $Z(p) \leq 2$ for all $p \leq 600$. We also obtain the congruence $p \mid \text{num}(a_{p-1})$ for all primes $p \geq 5$. All statements are verified exhaustively for $10 \leq n \leq 220$ (4984 prime–index pairs, zero exceptions). An appendix gives a second proof of the structure theorem via a Dwork-type *recurrence–seed* mechanism, proves the exact analogue of the structure theorem for the $\zeta(2)$ Apéry sequences unconditionally, and identifies a character twist (χ_{-3}, χ_{-4}) in the corresponding seed congruences for Zagier's sporadic families — a p -adic counterpart of Zagier's Eichler-integral description of the second solutions. We do not claim a proof of Conjecture 3.2; the results below are a structure theorem, an exact reduction, and unconditional/conditional partial results.

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1 Setup and notation

Following Problem 3.2 of the Ramanujan Challenge [18], let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be the two solutions of Apéry's recurrence

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5) u_n + n^3 u_{n-1} = 0 \quad (n \geq 1),$$

with initial values $a_0 = 0$, $a_1 = 6$ and $b_0 = 1$, $b_1 = 5$. The b_n are the Apéry numbers

$$b_n = \sum_{k=0}^n B_{n,k}, \quad B_{n,k} := \binom{n}{k}^2 \binom{n+k}{k}^2 \in \mathbb{Z}_{\geq 0}.$$

Set $d_n := \text{lcm}(1, \dots, n)^3$. By [2] one has $d_n a_n, d_n b_n \in \mathbb{Z}$. Write

$$g_n := \gcd(d_n a_n, d_n b_n),$$

so that Conjecture 3.2 is the statement $g_n = e^{o(n)}$, equivalently $\log g_n = o(n)$.

For a prime p we write v_p for the p -adic valuation on $\mathbb{Q}$, and for an integer m we write $m = \sum_i m_i p^i$ for its base- p expansion (so $m_0, m_1, \dots$ are the base- p digits). For a nonzero rational x we write $\text{num}(x), \text{denom}(x)$ for the numerator and denominator of x in lowest terms. Because $\sqrt{n} < p \leq n$ implies $n < p^2$, every integer in $[0, n]$ has at most two base- p digits, and every integer in $[0, 2n]$ has at most three. We repeatedly use Kummer's theorem [8]: $v_p \binom{x+y}{y}$ equals the number of carries when adding x and y in base p ; and Lucas' theorem: $\binom{x}{y} \equiv \prod_i \binom{x_i}{y_i} \pmod{p}$.

The explicit form of a_n we use throughout is the one of Apéry, made explicit by Beukers and van der Poorten [3, 15]:

$$a_n = \sum_{k=0}^n B_{n,k} c_{n,k}, \quad c_{n,k} = H_n^{(3)} + \sum_{m=1}^k \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}}, \quad (1)$$

where $H_n^{(3)} = \sum_{m=1}^n m^{-3}$. Interchanging the order of summation and writing

$$S_m := \sum_{k=m}^n B_{n,k},$$

equation (1) becomes

$$a_n = b_n H_n^{(3)} + \sum_{m=1}^n \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} S_m. \quad (2)$$

2 Reduction to large primes

Lemma 1 (small primes are negligible). $\sum_{p \leq \sqrt{n}} v_p(g_n) \log p \leq 3 \pi(\sqrt{n}) \log n = O(\sqrt{n} \log n) = o(n)$.

Proof. For any prime p , $v_p(g_n) = v_p(d_n) + \min(v_p(a_n), v_p(b_n)) \leq v_p(d_n) = 3 \lfloor \log_p n \rfloor \leq 3 \log_p n$, so $v_p(g_n) \log p \leq 3 \log n$. Summing over the $\pi(\sqrt{n})$ primes $p \leq \sqrt{n}$ gives the bound, and $\pi(\sqrt{n}) \log n = O(\sqrt{n} \log n) = o(n)$. $\square$

Lemma 2 (p -adic normal form). For $\sqrt{n} < p \leq n$ one has $v_p(d_n) = 3$, $v_p(b_n) \geq 0$, $v_p(a_n) \geq -3$, and

$$v_p(g_n) > 0 \iff v_p(a_n) > -3.$$

Proof. Since $\sqrt{n} < p \leq n$, exactly one multiple of p (namely p itself, and no higher power) lies in $[1, n]$, so $v_p(\text{lcm}(1, \dots, n)) = 1$ and $v_p(d_n) = 3$. As $b_n \in \mathbb{Z}$, $v_p(b_n) \geq 0$; as $d_n a_n \in \mathbb{Z} [2]$, $v_p(a_n) \geq -v_p(d_n) = -3$. Now

$$v_p(g_n) = v_p(d_n) + \min(v_p(a_n), v_p(b_n)) = 3 + \min(v_p(a_n), v_p(b_n)).$$

Because $v_p(b_n) \geq 0 > -3$, we have $\min(v_p(a_n), v_p(b_n)) > -3$ if and only if $v_p(a_n) > -3$. Hence $v_p(g_n) > 0 \iff 3 + \min(v_p(a_n), v_p(b_n)) > 0 \iff v_p(a_n) > -3$. $\square$

By Lemma 1 the entire content of Conjecture 3.2 lies in the primes $p \in (\sqrt{n}, n]$, and by Lemma 2 the question there is whether the p^{-3} -part of a_n vanishes.

3 The digit criterion

Fix $\sqrt{n} < p \leq n$ and write $n = jp + n_0$ with $0 \leq n_0 < p$ and $1 \leq j < p$ (so $j = \lfloor n/p \rfloor$). By Lemma 2, $p^3 a_n$ is p -integral, and we set

$$X_{n,p} := (p^3 a_n) \bmod p \in \mathbb{F}_p,$$

so that

$$v_p(a_n) > -3 \iff X_{n,p} \equiv 0 \pmod{p}. \quad (3)$$

3.1 Only the multiples of p survive

Lemma 3 (term-by-term valuation). *Fix $\sqrt{n} < p \leq n$. For every $m \in \{1, \dots, n\}$ with $p \nmid m$ and every $k \in \{m, \dots, n\}$, the term*

$$T(n, k, m) := \frac{(-1)^{m-1} B_{n,k}}{2m^3 \binom{n}{m} \binom{n+m}{m}}$$

of the double sum in (1) satisfies $v_p(T(n, k, m)) \geq -2$. Consequently these terms contribute nothing to $X_{n,p}$.

Proof. Since $p \nmid m$, $v_p(m^3) = 0$, so $v_p(T(n, k, m)) = v_p(B_{n,k}) - v_p\left(\binom{n}{m}\right) - v_p\left(\binom{n+m}{m}\right)$. By Kummer's theorem, $v_p\left(\binom{n}{m}\right)$ is the number of carries in the base- p addition $m + (n - m) = n$. Because $n < p^2$, a carry out of digit position 1 would create a nonzero digit in position 2, which n does not have; hence there is at most one carry (in position 0) and $v_p\left(\binom{n}{m}\right) \leq 1$. Similarly $n + m < 2p^2$ has at most three base- p digits, so the addition $n + m$ carries at most in positions 0 and 1 and $v_p\left(\binom{n+m}{m}\right) \leq 2$. Thus the “denominator deficiency” $D := v_p\left(\binom{n}{m}\right) + v_p\left(\binom{n+m}{m}\right)$ satisfies $D \leq 3$.

If $D \leq 2$, then $v_p(T) = v_p(B_{n,k}) - D \geq 0 - 2 = -2$, as claimed.

Suppose $D = 3$, i.e. $v_p\left(\binom{n}{m}\right) = 1$ and $v_p\left(\binom{n+m}{m}\right) = 2$. Reading off the carries:

- $v_p\left(\binom{n}{m}\right) = 1$ means a carry in position 0 of $m + (n - m) = n$, i.e. a borrow in position 0 of $n - m$, which is equivalent to $m_0 > n_0$;
- $v_p\left(\binom{n+m}{m}\right) = 2$ means carries in positions 0 and 1 of $n+m$, i.e. $n_0 + m_0 \geq p$ (position 0) and $n_1 + m_1 + 1 \geq p$ (position 1, including the carry from position 0).

We claim that then $v_p(B_{n,k}) \geq 2$ for every k with $m \leq k \leq n$. Suppose not, so that $B_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$ has $v_p(B_{n,k}) = 0$, i.e. both $\binom{n}{k}$ and $\binom{n+k}{k}$ are prime to p . Prime-to- p of $\binom{n}{k}$ forces (Lucas) $k_0 \leq n_0$ and $k_1 \leq n_1$. Since $m_0 > n_0 \geq k_0$ we have $m_0 > k_0$; combined with $m \leq k$ (a two-digit comparison) this forces $m_1 < k_1$, i.e. $k_1 \geq m_1 + 1$. Then

$$n_1 + k_1 \geq n_1 + m_1 + 1 \geq p,$$

which is a carry in position 1 of the addition $n+k$, contradicting $v_p\left(\binom{n+k}{k}\right) = 0$. Hence $v_p(B_{n,k}) \geq 2$ and $v_p(T) = v_p(B_{n,k}) - 3 \geq -1 \geq -2$.

In all cases $v_p(T) \geq -2$, so $v_p(p^3 T) \geq 1$ and $p^3 T \equiv 0 \pmod{p}$; these terms do not affect $X_{n,p}$. $\square$

By Lemma 3, in computing $X_{n,p}$ from (2) we may discard every correction term with $p \nmid m$; only the indices $m \in \{p, 2p, \dots, jp\}$ and the p -divisible part of $H_n^{(3)}$ survive. Writing $m = j'p$ ($1 \leq j' \leq j$) and $H_n^{(3)} = \sum_{m=1}^n m^{-3}$, the multiples of p in the harmonic sum give $\sum_{j'=1}^j (j'p)^{-3}$, whence $p^3 H_n^{(3)} \equiv \sum_{j'=1}^j j'^{-3} =: H_j^{(3)} \pmod{p}$. For the surviving correction terms, using $(-1)^{j'p-1} = (-1)^{j'-1}$ (p odd) and $S_{j'} := \sum_{k=j'p}^n B_{n,k}$,

$$X_{n,p} \equiv b_n H_j^{(3)} + \sum_{j'=1}^j \frac{(-1)^{j'-1}}{2j'^3} S_{j'} \left[\binom{n}{j'p} \binom{n+j'p}{j'p} \right]^{-1} \pmod{p}, \quad (4)$$

where the inverse is taken in $\mathbb{F}_p$ for those j' for which the two binomials are units, and the summand is understood to be 0 for those j' for which one of the binomials is divisible by p (Lemma 4 below shows such summands indeed vanish p -adically).

3.2 Vanishing of the carry terms

Lemma 4 (carry-vanishing; double Kummer). *Let $\sqrt{n} < p \leq n$, $n = jp + n_0$. For $k = ip + k_0$ with $k_0 \leq n_0$: if $n_0 + k_0 \geq p$ or $j + i \geq p$ then $B_{n,k} \equiv 0 \pmod{p^2}$. Consequently, for each $1 \leq j' \leq j$ with $j + j' \geq p$,*

$$\sum_{k \geq j'p} B_{n,k} \equiv 0 \pmod{p^2},$$

and the corresponding term of (4) contributes no p^{-3} part; the same statement holds verbatim for the Beukers expansion of a_j itself.

Proof. Consider the addition $n + k$ in base p , with $n = jp + n_0$, $k = ip + k_0$. Its digit-0 sum is $n_0 + k_0$ and its digit-1 sum is $j + i$ (plus any carry from position 0). If $n_0 + k_0 \geq p$ there is a carry in position 0; if $j + i \geq p$ there is a carry in position 1. In either case Kummer gives $v_p \binom{n+k}{k} \geq 1$, and since $B_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$ carries this factor squared, $v_p(B_{n,k}) \geq 2$.

For the consequence, $S_{j'} = \sum_{k \geq j'p} B_{n,k}$ runs over $k = ip + k_0$ with $i \geq j'$ (and $k_0 \leq n_0$, since $\binom{n}{k} = 0$ for $k_0 > n_0$). If $j + j' \geq p$ then every such k has $j + i \geq j + j' \geq p$, so $B_{n,k} \equiv 0 \pmod{p^2}$ termwise and $S_{j'} \equiv 0 \pmod{p^2}$. Now $\binom{n}{j'p}$ is a unit (the addition $j'p + (n - j'p) = n$ has digits $(j', 0) + (j - j', n_0)$ and carries nowhere), while $\binom{n+j'p}{j'p}$ has exactly the single carry $j + j' \geq p$ in position 1, so $v_p \binom{n+j'p}{j'p} = 1$. The corresponding summand

of (4) therefore has valuation $v_p(S_{j'}) - 0 - 1 \geq 2 - 1 = 1 > 0$ and is $\equiv 0 \pmod{p}$.

For a_j : its Beukers expansion (2) (with n replaced by j , all integers now $< p$) has, for the index $m = j'$ with $j + j' \geq p$, denominator $\binom{j+j'}{j'}$ with a single carry $j + j' \geq p$, hence $v_p\left(\binom{j+j'}{j'}\right) = 1$; and its numerator $\sum_{i=j'}^j B_{j,i}$ is $\equiv 0 \pmod{p^2}$ because each $B_{j,i}$ ($i \geq j'$) has the carry $j + i \geq p$ in $\binom{j+i}{i}$, so $v_p(B_{j,i}) \geq 2$. Thus the same indices j' drop out of both expansions. $\square$

3.3 Digit reduction

Lemma 5 (digit reduction). *For $\sqrt{n} < p \leq n$, the invariant $X_{n,p}$ satisfies*

$$X_{n,p} \equiv b_{n_0} \cdot a_j \pmod{p},$$

where $a_j \in \mathbb{Q}$ is p -integral (its denominator divides $\text{lcm}(1, \dots, j)^3$, and $j < p$).

Proof. We reduce (4) using three ingredients.

(i) *Gessel's congruence* [5]: $b_n \equiv b_j b_{n_0} \pmod{p}$.

(ii) *Lucas for the outer binomials*: $\binom{n}{j'p} \equiv \binom{j}{j'}$ and $\binom{n+j'p}{j'p} \equiv \binom{j+j'}{j'}$ $\pmod{p}$ (the low digit is 0 resp. n_0 and contributes a factor 1).

(iii) *Reduction of $S_{j'}$* : write $k = ip + k_0$. By Lucas applied to each binomial,

$$B_{n,k} \equiv \binom{j}{i}^2 \binom{n_0}{k_0}^2 \binom{j+i}{i}^2 \binom{n_0+k_0}{k_0}^2 = B_{j,i} B_{n_0,k_0} \pmod{p},$$

valid when there is no digit overflow. Terms with $k_0 > n_0$ vanish since $\binom{n_0}{k_0} = 0$. Terms with $n_0 + k_0 \geq p$ have, by Lemma 4, $B_{n,k} \equiv 0 \pmod{p^2}$, hence $\equiv 0 \pmod{p}$. Therefore, for j' in the surviving range $j + j' < p$,

$$S_{j'} = \sum_{k \geq j'p} B_{n,k} \equiv \left(\sum_{i=j'}^j B_{j,i} \right) \left(\sum_{\substack{0 \leq k_0 \leq n_0 \\ n_0 + k_0 < p}} B_{n_0,k_0} \right) \equiv \left(\sum_{i=j'}^j B_{j,i} \right) b_{n_0} \pmod{p},$$

the last step because the omitted terms k_0 with $n_0 + k_0 \geq p$ are all $\equiv 0 \pmod{p}$ (again Lemma 4), so the truncated inner sum equals the full sum $\sum_{k_0=0}^{n_0} B_{n_0,k_0} = b_{n_0}$. This truncation identity is precisely why the criterion holds even in the “carry band” $n_0 \geq p/2$.

Substituting (i)–(iii) into (4) and factoring out b_{n_0} ,

$$X_{n,p} \equiv b_{n_0} \left[b_j H_j^{(3)} + \sum_{j'=1}^j \frac{(-1)^{j'-1}}{2 j'^3} \left(\sum_{i=j'}^j B_{j,i} \right) \left[\binom{j}{j'} \binom{j+j'}{j'} \right]^{-1} \right] \pmod{p},$$

where the bracketed indices j' with $j+j' \geq p$ drop out of both this expression and of a_j by Lemma 4. Relabelling $j' \mapsto m$ and $i \mapsto k$, the bracket is term-by-term the Beukers expansion (2) of a_j :

$$a_j = b_j H_j^{(3)} + \sum_{m=1}^j \frac{(-1)^{m-1}}{2m^3} \left(\sum_{k=m}^j B_{j,k} \right) \left[\binom{j}{m} \binom{j+m}{m} \right]^{-1}.$$

Hence $X_{n,p} \equiv b_{n_0} a_j \pmod{p}$. Finally $\text{denom}(a_j) \mid \text{lcm}(1, \dots, j)^3$ (as $d_j a_j \in \mathbb{Z}$) and $j < p$, so a_j is p -integral and $a_j \pmod{p}$ is well-defined; the reduction $b_{n_0} a_j \pmod{p}$ is meaningful. As a numerical check, $a_1 = 6$ and $a_2 = 351/4$ match the direct evaluation of the bracket. $\square$

Remark 1. *The cleanest point of the argument is that the criterion survives the carry band $p \leq 2\sqrt{n}$ (where $j+j' \geq p$ can occur). Let us spell out why the two truncations coincide. For an index j' with $j+j' \geq p$: on the $X_{n,p}$ side, by Lemma 4 the corresponding summand of the expansion of a_n has p -adic valuation ≥ -2 , hence contributes nothing to the p^{-3} -coefficient $X_{n,p}$; on the a_j side, the corresponding summand of the expansion of a_j has valuation $\geq +1$, hence vanishes in $a_j \pmod{p}$. For an index j' with $j+j' < p$, the two summands are identified term-by-term by Lucas as in Lemma 5(iii). Thus every index $j' \in \{1, \dots, j\}$ is accounted for on both sides — either matched (when $j+j' < p$) or discarded by both reductions (when $j+j' \geq p$) — and the congruence $X_{n,p} \equiv b_{n_0} a_j$ requires no further case analysis. This matching was found numerically first (1905 carry-band pairs, zero exceptions; see Section 5) and the mechanism above is its proof.*

3.4 The structure theorem

Theorem 1 (structure theorem). *Let p be a prime with $\sqrt{n} < p \leq n$ and write $n = jp + n_0$, $0 \leq n_0 < p$. Then*

$$v_p(\gcd(d_n a_n, d_n b_n)) > 0 \iff p \mid b_{n_0} \text{ or } p \mid \text{num}(a_j).$$

Proof. By Lemma 2 and (3), $v_p(g_n) > 0 \iff X_{n,p} \equiv 0 \pmod{p}$. By Lemma 5, $X_{n,p} \equiv b_{n_0} a_j \pmod{p}$, and since a_j is p -integral, $X_{n,p} \equiv 0 \pmod{p} \iff p \mid b_{n_0} \cdot \text{num}(a_j) \iff p \mid b_{n_0} \text{ or } p \mid \text{num}(a_j)$. $\square$

The criterion is symmetric between the two Apéry solutions: the integer solution b is tested at the low base- p digit n_0 of n , while the rational solution a is tested at the high digit $j = \lfloor n/p \rfloor$. The band phenomenon $p \in (\sqrt{n}, 2\sqrt{n}]$ is exactly the appearance of the a -side condition $p \mid \text{num}(a_j)$; e.g. $41 \mid \text{num}(a_5)$ and $13 \mid \text{num}(a_{12})$ produce the first such bands.

3.5 A companion congruence

Proposition 1 (a companion congruence). *For every prime $p \geq 5$: $p \mid \text{num}(a_{p-1})$.*

Proof. By (2), $a_{p-1} = b_{p-1}H_{p-1}^{(3)} + \Phi_{p-1}$ with

$$\Phi_{p-1} = \sum_{m=1}^{p-1} \frac{(-1)^{m-1}}{2m^3} \left(\sum_{k=m}^{p-1} B_{p-1,k} \right) \left[\binom{p-1}{m} \binom{p-1+m}{m} \right]^{-1},$$

and a_{p-1} is p -integral ($\text{denom}(a_{p-1}) \mid \text{lcm}(1, \dots, p-1)^3$, prime to p).

First term. As m runs over $\{1, \dots, p-1\}$, m^{-1} runs over all of $\mathbb{F}_p^\times$, so $H_{p-1}^{(3)} = \sum_{m=1}^{p-1} m^{-3} \equiv \sum_{u \in \mathbb{F}_p^\times} u^3 \pmod{p}$. The power sum $\sum_{u \in \mathbb{F}_p^\times} u^3$ vanishes in $\mathbb{F}_p$ whenever $(p-1) \nmid 3$; for $p \geq 5$ we have $p-1 \geq 4 > 3$, so $(p-1) \nmid 3$ and $H_{p-1}^{(3)} \equiv 0 \pmod{p}$. Hence $b_{p-1}H_{p-1}^{(3)} \equiv 0 \pmod{p}$.

Correction term. Here $n = p-1$ has digits $n_0 = p-1$, $n_1 = 0$, and each m with $1 \leq m \leq p-1$ has $p \nmid m$. In $\binom{p-1+m}{m}$ the position-0 sum is $(p-1)+m \geq p$, so there is exactly one carry and $v_p\left(\binom{p-1+m}{m}\right) = 1$; while $\binom{p-1}{m}$ has digit sum $m + (p-1-m) = p-1 < p$ with no carry, so it is a unit. The numerator $\sum_{k=m}^{p-1} B_{p-1,k}$ has, for each $k \geq m \geq 1$, a carry $(p-1) + k \geq p$ in $\binom{p-1+k}{k}$, hence $v_p(B_{p-1,k}) \geq 2$ and the sum is $\equiv 0 \pmod{p^2}$. The m -th summand of Φ_{p-1} therefore has valuation $\geq 2 - 0 - 1 = 1 > 0$, so $\Phi_{p-1} \equiv 0 \pmod{p}$.

Thus $a_{p-1} \equiv 0 \pmod{p}$, i.e. $p \mid \text{num}(a_{p-1})$. $\square$

Remark 2. *By Theorem 1 (with $j = p-1$, i.e. $n \in [p^2-p, p^2)$), the windows containing a factor $p \mid \text{num}(a_{p-1})$ are always “bad”; but the relevant prime there is $p \sim \sqrt{n}$, of weight $O(\log n)$, hence harmless for Conjecture 3.2. A mod- p^2 refinement (via Glaisher-type congruences for $H_{p-1}^{(3)}$) may be possible but is not pursued here.*

4 Consequences for Problem 3.2

Corollary 1 (equivalent form of Conjecture 3.2). *Conjecture 3.2 holds if and only if*

$$\sum_{\substack{\sqrt{n} < p \leq n \\ p \mid b_n \bmod p \vee p \mid \text{num}(a_{\lfloor n/p \rfloor})}} v_p(g_n) \log p = o(n).$$

Proof. $\log g_n = \sum_p v_p(g_n) \log p$. By Lemma 1 the primes $p \leq \sqrt{n}$ contribute $O(\sqrt{n} \log n) = o(n)$ unconditionally. By Theorem 1, among primes $p \in (\sqrt{n}, n]$ the summand $v_p(g_n) > 0$ exactly on the indicated set. Hence $\log g_n = o(n)$ if and only if the displayed sum is $o(n)$. $\square$

Theorem 2 (*a-side: unconditional almost-all bound*). *For every $\varepsilon > 0$, the number of $n \in [N, 2N]$ with $\sum_{p: p \mid \text{num}(a_{\lfloor n/p \rfloor})} \log p > \varepsilon n$ is $O(\sqrt{N}/\varepsilon) = o(N)$.*

Proof. Write $S_{\text{II}}(n) = \sum_{p > \sqrt{N}: p \mid \text{num}(a_{\lfloor n/p \rfloor})} \log p$ (primes $p \leq \sqrt{N}$ contribute $o(n)$ by Lemma 1 and may be absorbed). We count pairs. Fix $j \geq 1$ and a prime $p > \sqrt{N}$ with $p \mid \text{num}(a_j)$; the integers $n \in [N, 2N]$ with $\lfloor n/p \rfloor = j$ form a subinterval of $[jp, (j+1)p)$ of length at most p , each contributing $\log p$. Moreover $p > \sqrt{N}$ and $jp \leq 2N$ force $j \leq 2N/p < 2\sqrt{N}$ and $p \leq 2N/j$. Hence

$$\sum_{n \in [N, 2N]} S_{\text{II}}(n) \leq \sum_{j \leq 2\sqrt{N}} \sum_{\substack{p \mid \text{num}(a_j) \\ \sqrt{N} < p \leq 2N/j}} p \log p \leq \sum_{j \leq 2\sqrt{N}} \frac{2N}{j} \log(2N) \omega_{>\sqrt{N}}(\text{num}(a_j)),$$

where $\omega_{>\sqrt{N}}$ counts distinct prime factors exceeding $\sqrt{N}$. Since $a_j = e^{O(j)}$ and $\text{denom}(a_j) \mid \text{lcm}(1, \dots, j)^3 = e^{O(j)}$, we have $\text{num}(a_j) = e^{O(j)}$, so $\omega_{>\sqrt{N}}(\text{num}(a_j)) \leq \log \text{num}(a_j) / \log \sqrt{N} = O(j / \log N)$. Therefore

$$\sum_{n \in [N, 2N]} S_{\text{II}}(n) \leq \sum_{j \leq 2\sqrt{N}} \frac{2N}{j} \log(2N) \cdot O\left(\frac{j}{\log N}\right) = \sum_{j \leq 2\sqrt{N}} O(N) = O(N^{3/2}).$$

By Markov's inequality, $\#\{n \in [N, 2N] : S_{\text{II}}(n) > \varepsilon n\} \leq \#\{n : S_{\text{II}}(n) > \varepsilon N\} \leq O(N^{3/2})/(\varepsilon N) = O(\sqrt{N}/\varepsilon) = o(N)$. $\square$

Theorem 3 (*b-side: conditional*). *If $Z(p) := \#\{0 \leq r < p : p \mid b_r\}$ satisfies $Z(p) = O(p / \log^{2+\varepsilon} p)$, then for almost all n*

$$\sum_{p > \sqrt{n}, v_p(g_n) > 0} \log p = o(n),$$

i.e. the squarefree support of g_n above $\sqrt{n}$ is $e^{o(n)}$ for almost all n .

Proof. By Theorem 1, for $p > \sqrt{n}$ the condition $v_p(g_n) > 0$ splits into the *a-side* ($p \mid \text{num}(a_{\lfloor n/p \rfloor})$) and the *b-side* ($p \mid b_{n \bmod p}$). Theorem 2 already

handles the a -side: all but $O(\sqrt{N}) = o(N)$ of the $n \in [N, 2N]$ have a -side contribution $\leq \varepsilon n$. For the b -side, put $S_I(n) = \sum_{p > \sqrt{N}: p | b_n \bmod p} \log p$. Then

$$\begin{aligned} \sum_{n \in [N, 2N]} S_I(n) &= \sum_{\sqrt{N} < p \leq 2N} \log p \cdot \#\{n \in [N, 2N] : p \mid b_n \bmod p\} \\ &= \sum_{\sqrt{N} < p \leq 2N} \log p \left(\frac{N}{p} + O(1) \right) Z(p), \end{aligned}$$

since among any p consecutive integers exactly $Z(p)$ lie in a residue class with $p \mid b_n \bmod p$. Under $Z(p) = O(p / \log^{2+\varepsilon} p)$,

$$\begin{aligned} \sum_p \frac{N}{p} Z(p) \log p &= O\left(N \sum_{\sqrt{N} < p \leq 2N} \frac{\log p}{\log^{2+\varepsilon} p}\right) = O\left(N \sum_{p \leq 2N} \frac{1}{\log^{1+\varepsilon} p}\right) \\ &= O\left(\frac{N^2}{\log^{2+\varepsilon} N}\right) = o(N^2), \end{aligned}$$

using $\sum_{p \leq x} \log^{-(1+\varepsilon)} p = O(x / \log^{2+\varepsilon} x)$; the $O(1)$ term contributes $O(\sum_{p \leq 2N} Z(p) \log p) = o(N^2)$ likewise, again because $Z(p) \log p = O(p / \log^{1+\varepsilon} p)$ under the hypothesis. Hence $\sum_{n \in [N, 2N]} S_I(n) = o(N^2)$, and by Markov $\#\{n \in [N, 2N] : S_I(n) > \varepsilon n\} = o(N)$. Combining the two sides, for all but $o(N)$ of $n \in [N, 2N]$ we have $\sum_{p > \sqrt{n}, v_p(g_n) > 0} \log p \leq 2\varepsilon n$; as $\varepsilon > 0$ is arbitrary and the ranges $[N, 2N]$ tile $\mathbb{N}$ dyadically, the bound $= o(n)$ holds for almost all n . $\square$

Remark 3. $Z(p) \leq 2$ for all $p \leq 600$ (computed; see Section 5); Malik–Straub’s heuristic (Conjecture 6.3 of [9]) predicts $Z(p) = O(1)$ on average with the exact non-divisibility density $e^{-1/2}$. No nontrivial upper bound on $Z(p)$ appears in the literature; the infinitude of $\{p : Z(p) = 0\}$ is an open question in the Rowland–Yassawi circle of ideas [12, 7]. Thus Theorem 3 localizes the b -side of Conjecture 3.2 in an established open problem, while Theorem 2 settles the a -side on average unconditionally. The structural expectation $Z(p) = O(1)$ is consistent with the interpretation of $b_r \bmod p$ as a fibre invariant of an algebraic family (via the modularity results of Beukers and Ahlgren–Ono [1]), but we do not have a proof.

5 Numerical verification

All computations were carried out in exact arithmetic (Python/`sympy`); the scripts are named below and are available upon request.

- **Structure theorem (Theorem 1).** Exhaustive check over all n with $10 \leq n \leq 220$ and all primes $p \in (\sqrt{n}, n]$: 4984 prime-index pairs, zero exceptions. Of these, 279 pairs are “bad” ($v_p(g_n) > 0$). The pairs split as 3079 in the no-carry band and 1905 in the carry band $p \in (\sqrt{n}, 2\sqrt{n}]$, and the criterion holds identically on both. Within the no-carry band the bad pairs split as 98 of a -side type and 67 of b -side type (`test_symmetric_rule.py`).
- **Intermediate stages.** The single-window criterion for $p \in (n/2, n]$ was checked for $10 \leq n \leq 260$: 3353 pairs, 100% agreement, 29 bad pairs (`test_criterion.py`). The general-window criterion (4) was checked for $10 \leq n \leq 220$: all 4984 pairs agree (`test_criterion_general.py`), including that the carry terms may be dropped (`verify_band_mechanism.py`). The top-window reduction to $p \mid b_{n-p}$ for $p \in (2n/3, n]$ was checked on 1633 pairs, zero mismatches (`test_apery_divis.py`).
- **Companion congruence (Proposition 1).** $p \mid \text{num}(a_{p-1})$ confirmed for all primes $5 \leq p \leq 59$ (`test_apery_divis.py`).
- $Z(p)$ (**Theorem 3, Remark**). For all 107 primes $p \leq 600$, $Z(p) \in \{0, 1, 2\}$; the average of $Z(p)/p$ is 0.012, with maximum $2/5$ at $p = 5$ (`measure_Zp.py`).
- **Growth.** $\log(g_n)/n$ was tabulated up to $n = 600$ (`explore_gcd.py`); it decays from about 0.24 at small n into the range 0.01–0.05, consistent with $\log g_n = o(n)$.

6 Open ends

We reiterate that Conjecture 3.2 is not proved here. What remains, in view of Corollary 1, is:

1. An unconditional bound on $Z(p)$ of the (very weak) form $O(p/\log^{2+\varepsilon} p)$ would upgrade Theorem 3 to unconditional; the computed values suggest $Z(p) = O(1)$, but no nontrivial bound is known.
2. Per- n (as opposed to almost-all) control of the a -side: one must exclude that the large prime factors of $\text{num}(a_j)$ systematically concentrate in the single window $(n/(j+1), n/j]$ that n selects. Adversarially this is not ruled out by our averaging, and appears to be the essential

obstruction; the conservative-matrix-field framework of Weinbaum et al. [16] may offer additional algebraic structure on $X_{n,p}$ here.

3. Multiplicities $v_p(g_n) \geq 2$: these require a mod- p^2 analysis of $X_{n,p}$ (a second-order digit criterion, e.g. via Lucas-type congruences mod p^2 in the sense of Rowland–Yassawi and Krattenthaler). The squarefree-support version of Theorem 3 is unaffected by this point.
4. A mod- p^2 refinement of Proposition 1 (Glaisher).

A The recurrence–seed mechanism: a second proof, the $\zeta(2)$ analogue, and character twists

Theorem 1 was proved above by a direct base- p analysis of the Beukers expansion. In this appendix we give a second, structurally different proof and put it to work twice: we prove the exact analogue of Theorem 1 for the $\zeta(2)$ Apéry sequences (Theorem 4), and we report and partly explain an arithmetic refinement in Zagier’s sporadic families, where the seed congruence constants turn out to be values of quadratic Dirichlet characters (Section A.4).

The mechanism has two halves.

Propagation. Write $n = jp + n_0$ as before and set $X(n_0) := p^3 a_{jp+n_0} \bmod p$. Apéry’s recurrence, multiplied by p^3 and reduced mod p , becomes a second-order linear recurrence for X in the low digit n_0 whose coefficients depend only on n_0 ; the comparison quantity $Y(n_0) := b_{n_0} a_j \bmod p$ satisfies the *same* recurrence. Crucially, the trailing coefficient n^3 of Apéry’s recurrence vanishes mod p exactly at the block boundary $n \equiv 0 \pmod{p}$, so mod p the recurrence degenerates to first order there: each block of p consecutive indices is, mod p , determined by its first value. Consequently a *single* seed congruence at $n_0 = 0$ suffices, and the second seed at $n_0 = 1$ is automatic. This is an elementary explanation of why Dwork-type block congruences (in the spirit of $F(x)/F(x^p)$) govern both solutions of the recurrence.

Seed. The seed at $n_0 = 0$ is a Dwork-type congruence, $p^3 a_{jp} \equiv a_j \pmod{p}$, which we prove from the Beukers expansion using Ljunggren’s binomial congruence. Ljunggren’s congruence is insensitive to base- p carries, so — unlike the digit analysis of Section 3 — no carry-band case distinction arises anywhere in this proof.

Throughout this appendix $p \geq 5$; the primes $p \in \{2, 3\}$ lie in $(\sqrt{n}, n]$ only for $n \leq 8$, where all statements are verified directly.

A.1 Two elementary lemmas

Lemma 6 (Ljunggren; see [6, §1]). *For every prime $p \geq 5$ and all integers $A \geq B \geq 0$,*

$$\binom{Ap}{Bp} \equiv \binom{A}{B} \pmod{p^3}.$$

No digit condition is required: the congruence holds (as a statement about the difference) also when $p \mid \binom{A}{B}$.

Lemma 7 (ratio lemma). *Let A, a, B, b be p -integral with $B, b \neq 0$, and suppose*

$$A \equiv a \pmod{p^2}, \quad B \equiv b \pmod{p^3}, \quad v_p(b) \leq 1, \quad v_p(a) \geq v_p(b).$$

Then A/B and a/b are p -integral and $A/B \equiv a/b \pmod{p}$.

Proof. From $B \equiv b \pmod{p^3}$ and $v_p(b) \leq 1 < 3$ we get $v_p(B) = v_p(b)$; from $A \equiv a \pmod{p^2}$ and $v_p(a) \geq v_p(b)$ we get $v_p(A) \geq \min(v_p(a), 2) \geq v_p(b)$, so both ratios are p -integral. Now

$$\frac{A}{B} - \frac{a}{b} = \frac{(A-a)b + a(b-B)}{Bb},$$

and $v_p((A-a)b) \geq 2 + v_p(b)$, $v_p(a(b-B)) \geq v_p(b) + 3$, while $v_p(Bb) = 2v_p(b)$. Hence the difference has valuation at least $\min(2 - v_p(b), 3 - v_p(b)) = 2 - v_p(b) \geq 1$. $\square$

A.2 Tail supercongruences and the seed for $\zeta(3)$

Lemma 8 (tail supercongruence). *Let $p \geq 5$, $1 \leq j \leq p-1$, $n = jp$, and $0 \leq t \leq j$. Then*

$$S_{tp} = \sum_{k \geq tp} B_{n,k} \equiv \sum_{i=t}^j B_{j,i} \pmod{p^2}.$$

Moreover, if $j+t \geq p$ then each term $B_{j,i}$ ($i \geq t$) is divisible by p^2 , so both sides are $\equiv 0 \pmod{p^2}$.

Proof. Split $k = ip + k_0$. If $k_0 \neq 0$, the base- p addition $k + (n-k) = n$ carries in position 0 (its digit-0 sum is $k_0 + (p-k_0) = p$), so by Kummer $v_p\binom{n}{k} \geq 1$ and $v_p(B_{n,k}) \geq 2$: these terms vanish mod p^2 . If $k = ip$, Lemma 6

gives $\binom{jp}{ip} \equiv \binom{j}{i}$ and $\binom{(j+i)p}{ip} \equiv \binom{j+i}{i} \pmod{p^3}$, hence (products of p -integral congruences mod p^3)

$$B_{n,ip} = \binom{jp}{ip}^2 \binom{(j+i)p}{ip}^2 \equiv \binom{j}{i}^2 \binom{j+i}{i}^2 = B_{j,i} \pmod{p^3}.$$

Summing over $i = t, \dots, j$ proves the congruence. For the last claim, if $j + i \geq j + t \geq p$ then the addition $j + i$ carries once, so $v_p\binom{j+i}{i} = 1$ and $v_p(B_{j,i}) \geq 2$. $\square$

Remark 4. The case $t = 0$ of Lemma 8 reads $b_{jp} \equiv b_j \pmod{p^2}$, a weak form of Beukers' supercongruence $b_{np} \equiv b_n \pmod{p^3}$ [4]. In particular the appendix is self-contained: it does not need Gessel's congruence.

Proposition 2 (seed congruence (T_0)). For $p \geq 5$ and $1 \leq j \leq p-1$:

$$p^3 a_{jp} \equiv a_j \pmod{p}.$$

Proof. Evaluate (2) at $n = jp$ and multiply by p^3 :

$$p^3 a_n = b_n (p^3 H_n^{(3)}) + \sum_{m=1}^n \frac{(-1)^{m-1}}{2} \cdot \frac{p^3}{m^3 \binom{n}{m} \binom{n+m}{m}} S_m.$$

Harmonic part. Splitting the sum at the multiples of p ,

$$p^3 H_{jp}^{(3)} = \sum_{t=1}^j t^{-3} + p^3 \sum_{m \leq jp, p \nmid m} m^{-3} = H_j^{(3)} + (\text{terms of valuation } \geq 3),$$

so, using $b_{jp} \equiv b_j \pmod{p}$ (Lemma 8, $t = 0$), $b_n(p^3 H_n^{(3)}) \equiv b_j H_j^{(3)} \pmod{p}$.

Terms with $p \nmid m$. Lemma 3 applies verbatim (it was proved for arbitrary n , in particular $n = jp$): every term of the double sum has valuation ≥ -2 , so after multiplication by p^3 it is $\equiv 0 \pmod{p}$.

Terms with $m = tp$, $1 \leq t \leq j$. Here

$$\frac{p^3}{(tp)^3 \binom{jp}{tp} \binom{(j+t)p}{tp}} S_{tp} = \frac{1}{t^3} \cdot \frac{S_{tp}}{\binom{jp}{tp} \binom{(j+t)p}{tp}},$$

and the sign is $(-1)^{tp-1} = (-1)^{t-1}$ (p odd). Apply Lemma 7 with

$$A = S_{tp}, \quad a = \sum_{i=t}^j B_{j,i}, \quad B = \binom{jp}{tp} \binom{(j+t)p}{tp}, \quad b = \binom{j}{t} \binom{j+t}{t}.$$

Indeed $A \equiv a \pmod{p^2}$ by Lemma 8; $B \equiv b \pmod{p^3}$ by Lemma 6 (product of two factors); $v_p(b) = v_p\left(\binom{j+t}{t}\right) = [j+t \geq p] \leq 1$ by Kummer ($\binom{j}{t}$ is a unit, $j < p$), where $[P] \in \{0, 1\}$ denotes the Iverson bracket; and $v_p(a) \geq 2[j+t \geq p] \geq v_p(b)$ by the last clause of Lemma 8. Hence

$$\frac{S_{tp}}{\binom{jp}{tp} \binom{(j+t)p}{tp}} \equiv \frac{\sum_{i \geq t} B_{j,i}}{\binom{j}{t} \binom{j+t}{t}} \pmod{p},$$

which is exactly the ratio occurring in the $m = t$ term of the expansion (2) of a_j (whose summation index m runs over $1, \dots, j < p$, so that expansion has no p -divisible indices at all). Summing over t ,

$$p^3 a_{jp} \equiv b_j H_j^{(3)} + \sum_{t=1}^j \frac{(-1)^{t-1}}{2t^3} \frac{\sum_{i \geq t} B_{j,i}}{\binom{j}{t} \binom{j+t}{t}} = a_j \pmod{p}. \quad \square$$

Note that no case distinction between the bands $j+t < p$ and $j+t \geq p$ was made: Lemma 6 does not see the carry, and Lemma 7 absorbs the common factor p that the carry inserts into numerator and denominator on both sides simultaneously.

Proposition 3 (boundary degeneration; the seed at $n_0 = 1$). *For $p \geq 5$ and $1 \leq j \leq p-1$:*

$$p^3 a_{jp+1} \equiv 5p^3 a_{jp} \equiv 5a_j = b_1 a_j \pmod{p}.$$

Proof. Multiply Apéry's recurrence at $n = jp$ by p^3 :

$$(jp+1)^3 [p^3 a_{jp+1}] = (34(jp)^3 + 51(jp)^2 + 27jp + 5) [p^3 a_{jp}] - (jp)^3 [p^3 a_{jp-1}].$$

All three bracketed quantities are p -integral ($jp+1 < p^2$, so $v_p(a_m) \geq -3$ for every index m involved). Reduce mod p : the leading coefficient is $\equiv 1$, the middle coefficient is $\equiv 5$, and the trailing term carries the factor $(jp)^3$ of valuation 3 and dies. The claim follows from Proposition 2. Finally $b_1 = 5$ is the same computation on the integer side: $1^3 b_1 = c(0) b_0$ with $c(n) = 34n^3 + 51n^2 + 27n + 5$, i.e. the constant 5 is $b_1 = c(0)$. $\square$

Proposition 4 (propagation; second proof of Lemma 5). *Let $p \geq 5$, $\sqrt{n} < p \leq n$, $n = jp + n_0$. Then $X(n_0) := p^3 a_{jp+n_0}$ and $Y(n_0) := b_{n_0} a_j$ satisfy $X(n_0) \equiv Y(n_0) \pmod{p}$ for all $0 \leq n_0 \leq p-1$; that is, $X_{n,p} \equiv b_{n_0} a_j \pmod{p}$.*

Proof. Multiplying the recurrence at index $m = jp + n_0$ by p^3 and reducing mod p gives, for $1 \leq n_0 \leq p - 2$,

$$(n_0 + 1)^3 X(n_0 + 1) \equiv c(n_0) X(n_0) - n_0^3 X(n_0 - 1) \pmod{p},$$

because the coefficients of Apéry's recurrence are polynomials in the index and hence depend mod p only on n_0 , and all quantities $p^3 a_{jp+n_0}$ ($0 \leq n_0 \leq p - 1$) are p -integral ($jp + n_0 \leq p^2 - 1$). The leading coefficient $(n_0 + 1)^3$ is invertible mod p for $n_0 \leq p - 2$, so the relation determines $X(n_0 + 1)$ from the two predecessors, for all $n_0 + 1 \leq p - 1$. The sequence $Y(n_0) = b_{n_0} a_j$ satisfies the identical relation: the integer recurrence for b at index $n_0 < p$, multiplied by the constant a_j . By Propositions 2 and 3, $X(0) \equiv Y(0)$ and $X(1) \equiv Y(1)$ ($b_0 = 1, b_1 = 5$); induction on n_0 finishes the proof. $\square$

Together with Lemma 2, Proposition 4 reproves Theorem 1 for $p \geq 5$ without any digit-by-digit analysis of the carry band.

Remark 5 (relation to Gessel's derivative term). *Gessel's mod- p^2 congruence $b_{pn+k} \equiv b_k b_n + pn \beta(k) b_n \pmod{p^2}$ [5], extended to all sporadic sequences by Straub [14], involves the formal derivative β of the integer solution, a p -adically integral object. The rational second solution a_n is a different object — its denominator is unbounded ($\text{denom}(a_n) \mid d_n$; already $a_2 = 351/4 \notin \mathbb{Z}$) — and the two must not be conflated. The congruences of this appendix concern the genuinely non-integral solution and appear not to be instances of the Gessel–Straub family.*

A.3 The $\zeta(2)$ analogue

Let now $(a'_n), (b'_n)$ be the two solutions of Apéry's $\zeta(2)$ recurrence

$$(n + 1)^2 u_{n+1} - (11n^2 + 11n + 3) u_n - n^2 u_{n-1} = 0,$$

with $a'_0 = 0, a'_1 = 5$ and $b'_0 = 1, b'_1 = 3$; the integer solution is $b'_n = \sum_k B'_{n,k}$ with $B'_{n,k} := \binom{n}{k}^2 \binom{n+k}{k}$, the sequence of Apéry's proof of the irrationality of $\zeta(2)$. The pair (a'_n, b'_n) plays the same role as before, now with $d'_n := \text{lcm}(1, \dots, n)^2$: one has $d'_n a'_n, d'_n b'_n \in \mathbb{Z}$, and we set $g'_n := \gcd(d'_n a'_n, d'_n b'_n)$. The analogue of the Beukers expansion (2) is

$$a'_n = b'_n \tilde{H}_n + \sum_{m=1}^n \frac{(-1)^{n+m-1}}{m^2 \binom{n}{m} \binom{n+m}{m}} S'_m, \quad (5)$$

where $\tilde{H}_n := 2 \sum_{m=1}^n (-1)^{m-1} m^{-2}$ and $S'_m := \sum_{k=m}^n B'_{n,k}$. It is obtained from the same construction that yields (1) (cf. [3, 15]); we have verified it

symbolically against the recurrence for all $n \leq 60$ (script `test_beukers_z2_extended.py`). Note the two structural differences from the $\zeta(3)$ case: the inner sign depends on n , and the binomial weight $B'_{n,k}$ contains the *first* power of $\binom{n+k}{k}$, so valuations of the summands can be odd. The latter is precisely what breaks a naive transfer of the digit analysis of Section 3 (the “double Kummer” Lemma 4 uses the squares); the recurrence–seed route is insensitive to it.

Theorem 4 ($\zeta(2)$ structure theorem). *Let $n \geq 2$ and let p be a prime with $\sqrt{n} < p \leq n$, $n = jp + n_0$. Then*

$$v_p(\gcd(d'_n a'_n, d'_n b'_n)) > 0 \iff p \mid b'_{n_0} \text{ or } p \mid \text{num}(a'_j).$$

The proof follows the $\zeta(3)$ pattern exactly; we state the three ingredients whose proofs differ, then assemble.

Lemma 9 (normal form). *For $\sqrt{n} < p \leq n$: $v_p(d'_n) = 2$, $v_p(a'_n) \geq -2$, and $v_p(g'_n) > 0 \iff v_p(a'_n) > -2 \iff X'(n_0) := p^2 a'_{jp+n_0} \equiv 0 \pmod{p}$.*

Proof. Identical to Lemma 2 with 3 replaced by 2. $\square$

Lemma 10 (tails, $\zeta(2)$). *Let $p \geq 5$, $1 \leq j \leq p-1$, $n = jp$, $0 \leq t \leq j$. Then $S'_{tp} \equiv \sum_{i=t}^j B'_{j,i} \pmod{p^2}$, and if $j+t \geq p$ then every $B'_{j,i}$ ($i \geq t$) is divisible by p (first power now), so $v_p(\sum_{i \geq t} B'_{j,i}) \geq 1$.*

Proof. As in Lemma 8: for $k = ip + k_0$, $k_0 \neq 0$, the addition $k + (n - k)$ carries in position 0, so $v_p\binom{jp}{k} \geq 1$ and already $v_p\left(\binom{jp}{k}^2\right) \geq 2$; for $k = ip$, Lemma 6 applied to the three binomial factors gives $B'_{jp,ip} \equiv B'_{j,i} \pmod{p^3}$. The carry statement is Kummer applied to $\binom{j+i}{i}$, which now enters to the first power. $\square$

Lemma 11 (terms with $p \nmid m$ at the seed). *Let $p \geq 5$, $n = jp$, and $1 \leq m \leq n$ with $p \nmid m$, $m = m_1 p + m_0$ ($m_0 \neq 0$). Then*

$$\frac{p^2}{m^2 \binom{n}{m} \binom{n+m}{m}} S'_m \equiv 0 \pmod{p}.$$

Proof. The two denominator valuations are *exact*: $v_p\binom{jp}{m} = 1$ (the addition $m + (jp - m)$ carries in position 0 since $m_0 + (p - m_0) = p$, and not in position 1 since $m_1 + (j - 1 - m_1) + 1 = j < p$), and $v_p\binom{jp+m}{m} = [j + m_1 \geq p]$ (position 0: $0 + m_0 < p$; position 1: $j + m_1$). If $j + m_1 < p$ the displayed quantity has valuation $\geq 2 - 0 - 1 - 0 = 1$ and we are done, S'_m being p -integral. If

$j + m_1 \geq p$ the prefactor has valuation 0 and we must show $p \mid S'_m$. Modulo p , the terms of S'_m with $k_0 \neq 0$ vanish (Lemma 10); the surviving terms are $k = ip$ with $ip \geq m$, i.e. $i \geq m_1 + 1$, and for those $B'_{jp,ip} \equiv B'_{j,i} \pmod{p}$ with $j + i \geq j + m_1 + 1 > p$, whence $p \mid \binom{j+i}{i}$ and $B'_{j,i} \equiv 0 \pmod{p}$ termwise. Hence $S'_m \equiv 0 \pmod{p}$. $\square$

Proposition 5 (seed and propagation, $\zeta(2)$). *For $p \geq 5$, $1 \leq j \leq p - 1$, $0 \leq n_0 \leq p - 1$:*

$$(S_0) \quad p^2 a'_{jp} \equiv a'_j, \quad (S_1) \quad p^2 a'_{jp+1} \equiv 3a'_j, \quad X'(n_0) \equiv b'_{n_0} a'_j \pmod{p}.$$

Proof. (S_0) : evaluate (5) at $n = jp$, multiply by p^2 . Harmonic part: $p^2 \tilde{H}_{jp} = 2 \sum_{t \leq j} (-1)^{pt-1} t^{-2} + p^2 \sum_{p \nmid m} (\dots) = \tilde{H}_j + (\text{valuation} \geq 2)$, using $(-1)^{pt-1} = (-1)^{t-1}$; and $b'_{jp} \equiv b'_j \pmod{p}$ (Lemma 10, $t = 0$). Terms $p \nmid m$: Lemma 11. Terms $m = tp$: the sign is $(-1)^{jp+tp-1} = (-1)^{j+t-1}$, and Lemma 7 applies with $A = S'_{tp}$, $a = \sum_{i \geq t} B'_{j,i}$, $B = \binom{jp}{tp} \binom{j+t}{tp}^p$, $b = \binom{j}{t} \binom{j+t}{t}$: the hypotheses hold by Lemmas 10 and 6, with $v_p(b) = [j + t \geq p] \leq 1$ and $v_p(a) \geq [j + t \geq p] = v_p(b)$. Each surviving term is thus $\equiv$ the $m = t$ term of the expansion (5) of a'_j , whose sign is $(-1)^{j+t-1}$ as well. Summing gives (S_0) .

(S_1) : multiply the recurrence at $n = jp$ by p^2 ; mod p the leading coefficient $(jp + 1)^2 \equiv 1$, the middle one $11(jp)^2 + 11jp + 3 \equiv 3$, and the trailing term carries $(jp)^2$ of valuation 2. Note $3 = b'_1 = c'(0)$.

Propagation: as in Proposition 4, with $(n_0 + 1)^2$ invertible for $n_0 \leq p - 2$ and $Y(n_0) = b'_{n_0} a'_j$ satisfying the same mod- p recurrence. Seeds (S_0) , (S_1) and induction give the digit reduction. $\square$

Proof of Theorem 4. For $p \geq 5$: Lemma 9 and Proposition 5, with a'_j p -integral ($j < p$), give $v_p(g'_n) > 0 \iff p \mid b'_{n_0} a'_j \iff p \mid b'_{n_0}$ or $p \mid \text{num}(a'_j)$. The finitely many pairs with $p \in \{2, 3\}$ (all have $n \leq 8$) are checked directly (`test_beukers_z2_extended.py`). $\square$

Remark 6. *The consequences of Section 4 transfer verbatim: the $\zeta(2)$ analogue of Conjecture 3.2 is equivalent to the sparsity of $\{p : p \mid b'_{n_0} \text{ or } p \mid \text{num}(a'_j)\}$; the a-side almost-all bound (Theorem 2) holds unconditionally with the same proof ($\text{num}(a'_j) = e^{O(j)}$); and the b-side reduces to bounds on $Z'(p) := \#\{0 \leq r < p : p \mid b'_r\}$. One new feature appears: $Z'(p)$ has structural zeros — $p \mid b'_{(p-1)/2}$ for every prime $p \equiv 3 \pmod{4}$ [13] — so $Z'(p) \geq 1$ for half the primes, and indeed we measure the mean of $Z'(p)$ to be 1.78 over $p \equiv 3 \pmod{4}$ versus 0.64 over $p \equiv 1 \pmod{4}$ ($p \leq 600$). For fixed n , however, the structural zero bites only when $n \equiv (p-1)/2 \pmod{p}$,*

i.e. $p \mid 2n + 1$, and $2n + 1$ has at most $O(1)$ prime factors exceeding $\sqrt{n}$; so the structural zeros are harmless for the $\zeta(2)$ analogue of Conjecture 3.2, and the essential difficulty is the same $Z'(p)$ problem as in Theorem 3.

A.4 Character twists in Zagier's sporadic families

Zagier [17] classified the recurrences

$$(n + 1)^2 u_{n+1} = (an^2 + an + b) u_n - cn^2 u_{n-1}$$

possessing an integral solution; besides degenerate and hypergeometric cases there are six sporadic families,

$$\begin{array}{lll} \mathbf{A} (7, 2, -8), & \mathbf{B} (9, 3, 27), & \mathbf{C} (10, 3, 9), \\ \mathbf{D} (11, 3, -1), & \mathbf{E} (12, 4, 32), & \mathbf{F} (17, 6, 72), \end{array}$$

(**D** being the $\zeta(2)$ pair of Section A.3). For each family let (B_n) be the integer solution ($u_0 = 1, u_1 = b$) and (A_n) the second solution ($u_0 = 0, u_1 = 1$); in all six families $\text{lcm}(1, \dots, n)^2 A_n \in \mathbb{Z}$ (verified for $n \leq 200$), i.e. $\kappa = 2$.

Numerically (all 92 pairs (j, p) with $p \in \{5, \dots, 37\}$, $1 \leq j < p$, $jp + 1 \leq 254$, and $p \nmid \text{num}(A_j)$; zero exceptions; `sporadic_seeds.py`) the seed congruences hold in the twisted form

$$p^2 A_{jp} \equiv C_0(p) A_j, \quad p^2 A_{jp+1} \equiv C_1(p) A_j \pmod{p}, \quad C_1 = b C_0,$$

with

$$C_0(p) = \begin{cases} 1, & \mathbf{A}, \mathbf{D}, \\ \chi_{-3}(p), & \mathbf{B}, \mathbf{C}, \mathbf{F}, \\ \chi_{-4}(p), & \mathbf{E}, \end{cases}$$

where χ_{-3}, χ_{-4} are the quadratic characters of conductors 3 and 4. Two remarks are in order.

The identity $C_1 = b C_0$ is a theorem, by the boundary degeneration of Proposition 3: multiplying the recurrence at $n = jp$ by p^2 and reducing mod p kills the trailing term (factor $(jp)^2$) and reduces the middle coefficient to $c'(0) = b$, so $p^2 A_{jp+1} \equiv b \cdot p^2 A_{jp} \pmod{p}$ for *every* family, whatever the value of C_0 .

The character matches Zagier's archimedean classification. Zagier [17, §3] describes the generating functions of the second solutions as Eichler integrals of weight-3 Eisenstein series twisted by precisely these characters,

the associated special values being rational multiples of $\zeta(2)$ (families **A**, **D**), $L(\chi_{-3}, 2)$ (families **C**, **F**, and, with complex characteristic roots and hence an oscillating rather than convergent Apéry limit, family **B**), and $L(\chi_{-4}, 2)$ (family **E**). The mod- p constants $C_0(p)$ reproduce exactly this partition of the six families by discriminant. The seed congruence thus appears to be a p -adic counterpart of Zagier’s Eichler-integral description — the same character seen at the real place (as an L -value) and at the finite places (as the value $\chi(p)$).

To our knowledge these congruences for the non-integral solutions are new. The extensive literature on congruences for Apéry-like numbers — Gessel [5], Beukers [4], Malik–Straub [9], Straub [14], and the Dwork-crystal framework — concerns the integer solutions only, and Osburn–Straub [10], who extended Zagier’s L -value correspondence to all six families, explicitly note the absence of p -adic congruences for the interpolated companion sequences.¹

Combining the (proved) Lucas congruences for the integer solutions [9] with the propagation principle, the twisted seed congruence is the *only* missing ingredient for the following:

Conjecture 1. *For each of the six sporadic families and every prime $p \geq 5$ with $\sqrt{n} < p \leq n$, $n = jp + n_0$:*

$$v_p(\gcd(d'_n A_n, d'_n B_n)) > 0 \iff p \mid B_{n_0} \text{ or } p \mid \text{num}(A_j).$$

Since $|C_0(p)| = 1$ in $\mathbb{F}_p^\times$, the twist does not affect the criterion. For the family **D** this is Theorem 4; for the others what remains is the analogue of Proposition 5(S_0), i.e. a Beukers-type explicit formula and its seed evaluation, from which the twist $C_0(p) = \chi(p)$ should emerge from the character of the associated Eisenstein series.

A.5 Numerical verification (appendix)

- **Seeds.** (T_0), (T_1), (S_0), (S_1) verified for all 85 pairs (j, p) , $p \leq 59$, $jp + 1 \leq 160$, both $\zeta(3)$ and $\zeta(2)$: zero failures (`test_seeds.py`, `test_jacobsthal_route.py`).
- **Ingredients.** Lemma 6 (441 cases), Lemma 10 (441 cases, both bands), Lemma 11 (556 cases), and the recurrence step of Propositions 3/ 5(S_1) (85 cases): zero failures (`test_jacobsthal_route.py`).

¹The “comparison of two rational approximations to $\zeta(3)$ ” of Osburn–Straub–Zudilin [11] is a different phenomenon: it compares Apéry’s b_n with another Apéry-like integer sequence, not the two solutions of one recurrence.

- **Theorem 4.** Exhaustive over $10 \leq n \leq 200$ and all $\sqrt{n} < p \leq n$: 4174 prime-index pairs, zero exceptions (`explore_zeta2.py`); the sharper identity $X'(n_0) \equiv b'_{n_0} a'_j$ on all 1047 not-bad pairs (`script verify_zeta2_structure.py`); the expansion (5) symbolically for $n \leq 60$ and the $p \in \{2, 3\}$ cases (`script test_beukers_z2_extended.py`).
- **Sporadic families.** 92 pairs per family, six families, with the stated characters: zero exceptions; $\kappa = 2$ with multiplier 1 in all families for $n \leq 200$ (`sporadic_seeds.py`).

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