

A Casoratian route to Dwork-type congruences for the second solutions of Apéry-like recurrences

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Abstract

For each of Zagier's six sporadic Apéry-like recurrences $(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}$ let (B_n) be the integer solution and (A_n) the rational second solution, normalised by $A_0 = 0$, $A_1 = 1$. Numerically the second solution satisfies a Dwork-type seed congruence $p^2 A_{jp} \equiv C_0(p) A_j \pmod{p}$ in which the constant $C_0(p)$ is a value of a quadratic Dirichlet character (1 , χ_{-3} or χ_{-4} according to the family). Our main observation is that this constant has an elementary closed form valid in *all six* families simultaneously,

$$C_0(p) \equiv B_{p-1}^{-1} \pmod{p},$$

so that the character twist is not an independent mystery but is entirely governed by the single residue $B_{p-1} \pmod{p}$ of the integer solution. We give a self-contained Casoratian derivation of the seed congruence: a universal formula for A_n/B_n as a telescoping sum, two vanishing lemmas for the zeros of B modulo p , an unconditional proof of the block-0 seed, and a shift lemma that transports the seed across every block whose left endpoint is a unit. Combined with the Malik–Straub Lucas congruences for B and with a propagation principle this yields, for all six families, the prime structure theorem $v_p(\gcd(d_n A_n, d_n B_n)) > 0 \iff p \mid B_{n_0} \text{ or } p \mid \text{num}(A_j) \text{ } (n = jp + n_0, \sqrt{n} < p \leq n)$, unconditionally at every prime with no zero of B in $[0, p)$ and, in general, modulo one explicitly isolated hypothesis (V6) concerning a two-step cross-Casoratian at a residue immediately following a zero of B . We further prove the character identity $B_{p-1} \equiv \chi(p)$

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(mod p) for five of the six families: for the trivial-character families it reduces by Lucas' theorem to a one-line computation (recovering, modulo p , classical congruences of Chowla–Cowles–Gessel and Beukers), and for three of the four character families it reduces to truncated-hypergeometric congruences of Sun–Tauraso and of Mortenson (Rodriguez-Villegas conjectures); for the remaining family it appears to be new. All statements are verified exhaustively: over 2380 prime-index pairs, zero exceptions.

1 Setup and notation

Following Zagier [15], consider the Apéry-like recurrence in normal form

$$(n+1)^\kappa u_{n+1} = c(n) u_n - \lambda n^\kappa u_{n-1} \quad (n \geq 1), \quad (1)$$

with $\kappa = 2$, $c(n) = an^2 + an + b \in \mathbb{Z}[n]$ and $\lambda = c$; here (a, b, c) runs over the six *sporadic* triples for which (1) admits an integer solution. We denote by $(B_n)_{n \geq 0}$ the integer solution normalised by $B_0 = 1$, $B_1 = b$, and by $(A_n)_{n \geq 0}$ the second solution normalised by $A_0 = 0$, $A_1 = 1$. In every sporadic family $\text{lcm}(1, \dots, n)^\kappa A_n \in \mathbb{Z}$ (verified for $n \leq 200$), i.e. the denominator growth exponent is $\kappa = 2$; we keep the letter κ in the formulas, as the Casoratian argument is insensitive to its value.

Family	(a, b, c)	$b = B_1$	$\lambda = c$	χ	OEIS
A	$(7, 2, -8)$	2	-8	1	A000172 (Franel)
B	$(9, 3, 27)$	3	27	χ_{-3}	A006077
C	$(10, 3, 9)$	3	9	χ_{-3}	A002893
D	$(11, 3, -1)$	3	-1	1	A005258 ($\zeta(2)$)
E	$(12, 4, 32)$	4	32	χ_{-4}	A081085
F	$(17, 6, 72)$	6	72	χ_{-3}	A093388

Throughout, $p \geq 5$ is a prime and v_p is the p -adic valuation on $\mathbb{Q}$. Two conditions that enter the arguments below are automatic in this range: $p \nmid \lambda$, because in every sporadic family λ is $\{2, 3\}$ -smooth $(-8, 27, 9, -1, 32, 72)$; and $p \nmid B_1 = b$, because $b \in \{2, 3, 4, 6\}$. (The prime divisors of λ contain the conductor of the character χ of the family, so nothing arithmetic is lost by $p \geq 5$.) The only genuine hypothesis in our main results is $p \nmid B_{p-1}$; by (12) below this is the statement $B_{p-1} \equiv \chi(p) \not\equiv 0$, which we prove for five of the six families in Section 9. We write $n = jp + n_0$ with $0 \leq n_0 < p$ and $1 \leq j < p$ for a prime $\sqrt{n} < p \leq n$, so that every integer in $[0, n]$ has at

most two base- p digits. We use repeatedly Fermat's congruence $\lambda^{p-1} \equiv 1 \pmod{p}$ (equivalently $\lambda^{tp} \equiv \lambda^t$) and the Lucas-type congruences for the integer solution, established for all sporadic sequences by Malik–Straub [6] and Straub [12]:

$$B_{tp+s} \equiv B_t B_s \pmod{p} \quad (0 \leq s < p), \quad (2)$$

of which we need the two instances $B_{tp} \equiv B_t$ and $B_{tp-1} \equiv B_{t-1} B_{p-1} \pmod{p}$.

The character values below were first found numerically (see Section 11 and [14, App.]) and are proved, for five of the six families, in Section 9: writing $C_0(p)$, $C_1(p)$ for the constants in the seed congruences $p^\kappa A_{jp} \equiv C_0(p) A_j$, $p^\kappa A_{jp+1} \equiv C_1(p) A_j \pmod{p}$, one has $C_1 = b C_0$ and

$$C_0(p) = \begin{cases} 1, & \mathbf{A}, \mathbf{D}, \\ \chi_{-3}(p), & \mathbf{B}, \mathbf{C}, \mathbf{F}, \\ \chi_{-4}(p), & \mathbf{E}, \end{cases} \quad (3)$$

χ_{-3}, χ_{-4} being the quadratic characters of conductors 3 and 4. The purpose of this note is to prove that $C_0(p) \equiv B_{p-1}^{-1} \pmod{p}$ in *all* six families and to derive the seed congruence directly, without a family-by-family Beukers-type explicit formula.

2 The universal Casoratian formula

Let $W_n := A_{n+1} B_n - A_n B_{n+1}$ be the Casoratian of the two solutions.

Lemma 1 (universal formula). $W_n = \lambda^n / (n+1)^\kappa$, and for all $n \geq 0$

$$\frac{A_n}{B_n} = \sum_{k=0}^{n-1} T_k, \quad T_k := \frac{\lambda^k}{(k+1)^\kappa B_k B_{k+1}}. \quad (4)$$

Proof. From (1) applied to both A and B , $(n+1)^\kappa W_n = (n+1)^\kappa (A_{n+1} B_n - A_n B_{n+1}) = \lambda n^\kappa (A_n B_{n-1} - A_{n-1} B_n) = \lambda n^\kappa W_{n-1}$, so $W_n = (\lambda n^\kappa / (n+1)^\kappa) W_{n-1}$. With $W_0 = A_1 B_0 - A_0 B_1 = 1$ this telescopes to $W_n = \lambda^n (n!)^\kappa / ((n+1)!)^\kappa = \lambda^n / (n+1)^\kappa$. Dividing $W_k = A_{k+1} B_k - A_k B_{k+1}$ by $B_k B_{k+1}$ gives $A_{k+1}/B_{k+1} - A_k/B_k = W_k / (B_k B_{k+1}) = T_k$; summing from $A_0/B_0 = 0$ yields (4). $\square$

Formula (4) holds in every family and requires no Beukers-type explicit form of A_n . It presupposes $B_k \neq 0$ (in $\mathbb{Q}$) for the indices involved; this holds

throughout the verified range $n \leq 300$ in all six families (families **A**, **C**, **D**, **E**, **F** have positive binomial representations, and the sign-alternating family **B** does not vanish there), and only the finitely many indices $k \leq p$ enter each proof below. The only defect of (4) is that the individual summands T_k have poles at the zeros of B modulo p ; the next two lemmas control them.

3 Zeros of the integer solution modulo p

Let $Z(p) := \{0 < r < p : p \mid B_r\}$.

Lemma 2 (zero-merge). *Assume $p \nmid B_{p-1}$. If $p \mid B_r$ with $0 < r < p$, then the two summands of (4) adjacent to B_r combine into a single p -integral expression free of B_r :*

$$T_{r-1} + T_r = \frac{\lambda^{r-1} c(r)}{r^\kappa (r+1)^\kappa B_{r-1} B_{r+1}}.$$

In particular the pole at B_r cancels regardless of the depth $v_p(B_r) \geq 1$.

Proof. Factor B_r out of both summands:

$$T_{r-1} + T_r = \frac{\lambda^{r-1}}{B_r} \left[\frac{1}{r^\kappa B_{r-1}} + \frac{\lambda}{(r+1)^\kappa B_{r+1}} \right] = \frac{\lambda^{r-1}}{B_r} \cdot \frac{(r+1)^\kappa B_{r+1} + \lambda r^\kappa B_{r-1}}{r^\kappa (r+1)^\kappa B_{r-1} B_{r+1}}.$$

By the recurrence (1) at index $n = r$, $(r+1)^\kappa B_{r+1} + \lambda r^\kappa B_{r-1} = c(r) B_r$, and the factor B_r cancels, leaving the stated p -integral expression (its denominator $r^\kappa (r+1)^\kappa B_{r-1} B_{r+1}$ is a p -unit once B_{r-1}, B_{r+1} are, which Lemma 3 guarantees). $\square$

Lemma 3 (non-adjacency). *If $p \nmid B_{p-1}$, then no two elements of $Z(p)$ are consecutive, and $Z(p) \subseteq \{1, \dots, p-2\}$; in particular for each $r \in Z(p)$ both B_{r-1} and B_{r+1} are p -units (for $r = 1$ this uses $B_0 = 1$). If moreover $p \nmid B_1$ — automatic for the six sporadic families and $p \geq 5$ — then $Z(p) \subseteq \{2, \dots, p-2\}$.*

Proof. Suppose $p \mid B_r$ and $p \mid B_{r+1}$ for some $r \leq p-3$. Reducing (1) at index $n = r+1$ modulo p gives $(r+2)^\kappa B_{r+2} \equiv c(r+1) B_{r+1} - \lambda (r+1)^\kappa B_r \equiv 0 \pmod{p}$; as $r+2 \leq p-1$ the factor $(r+2)^\kappa$ is invertible, so $p \mid B_{r+2}$. Iterating forward gives $p \mid B_{p-1}$, contradicting the hypothesis. Thus zeros are non-adjacent. The value $B_0 = 1$ and (by hypothesis) B_{p-1} are units, so $Z(p)$ avoids 0 and $p-1$; non-adjacency then makes both neighbours of every $r \in Z(p)$ units. $\square$

4 The block-0 seed

Proposition 1 (block-0 seed, unconditional). *Let $p \geq 5$ with $p \nmid \lambda B_{p-1}$. Then*

$$p^\kappa A_p \equiv B_{p-1}^{-1} = B_{p-1}^{-1} A_1 \pmod{p}.$$

Proof. By Lemma 1, $p^\kappa A_p = B_p \sum_{k=0}^{p-1} p^\kappa T_k$. We isolate the unique summand whose valuation is negative. For $k \leq p-2$ the factor $(k+1)^\kappa$ is a p -unit; grouping the (by Lemma 3 non-adjacent, and away from 0 and $p-1$) zeros $r \in Z(p)$ through Lemma 2, the partial sum $\sum_{k=0}^{p-2} T_k$ is p -integral. Since $B_p \in \mathbb{Z}$, multiplication by p^κ annihilates this part: $p^\kappa B_p \sum_{k=0}^{p-2} T_k \equiv 0 \pmod{p}$.

In the remaining summand $k = p-1$ — the only pole of order κ — the factor B_p cancels exactly:

$$p^\kappa B_p T_{p-1} = p^\kappa B_p \cdot \frac{\lambda^{p-1}}{p^\kappa B_{p-1} B_p} = \frac{\lambda^{p-1}}{B_{p-1}} \equiv B_{p-1}^{-1} \pmod{p},$$

using Fermat's congruence $\lambda^{p-1} \equiv 1$ and $p \nmid B_{p-1}$. Hence $p^\kappa A_p \equiv B_{p-1}^{-1}$, and $A_1 = 1$. $\square$

This is the seed congruence (3) in the case $j = 1$, and it already identifies the constant: $C_0(p) \equiv B_{p-1}^{-1} \pmod{p}$. The remaining sections propagate it to $1 < j < p$.

5 The shift lemma

For $0 \leq u < p-1$ set

$$G_u := A_{(u+1)p} B_{up} - A_{up} B_{(u+1)p}.$$

Lemma 4 (shift lemma). *Let $p \geq 5$ with $p \nmid \lambda B_{p-1}$. Then for all $0 \leq u < p-1$*

$$p^\kappa G_u \equiv \frac{\lambda^u}{(u+1)^\kappa B_{p-1}} \pmod{p}.$$

Proof. The argument transports the block-0 seed of Proposition 1 to the block $[up, (u+1)p)$ without ever dividing by B_u ; the difficulty caused by a zero of B inside the block is absorbed into the observation that a suitably restarted second solution is congruent to the original one modulo p .

Step 1 (a solution vanishing at up). Fix u and set $D(x) := A_x B_{up} - A_{up} B_x$. As a fixed linear combination of the two solutions of (1), D is itself a solution in the variable x . It satisfies $D(up) = 0$ and

$$D(up+1) = A_{up+1} B_{up} - A_{up} B_{up+1} = W_{up} = \frac{\lambda^{up}}{(up+1)^\kappa},$$

a p -unit value (the denominator $up+1$ is a p -unit).

Step 2 (factorisation through a restarted solution). Let $(\tilde{u}_x)_{x \geq 0}$ be the solution of the recurrence obtained by restarting at index up , i.e. with the same coefficients $c(up+x)$, $(up+x)^\kappa$ and initial data $\tilde{u}_0 = 0$, $\tilde{u}_1 = 1$. Because a second-order linear recurrence with nonzero (rational) leading coefficient has a unique solution with prescribed $\tilde{u}_0, \tilde{u}_1$, and $D(up) = 0$, $D(up+1) = W_{up}$, we get the identity $D(up+x) = W_{up} \tilde{u}_x$ for all $x \geq 0$. In particular

$$G_u = D((u+1)p) = W_{up} \tilde{u}_p. \quad (5)$$

Step 3 (the restarted solution equals $A \bmod p$). For $0 \leq x \leq p-1$ the value $\tilde{u}_x$ is p -integral: by induction the divisor $(up+x+1)^\kappa$ occurring at step x is a p -unit for $x \leq p-2$. Moreover, modulo p ,

$$c(up+x) \equiv c(x), \quad (up+x)^\kappa \equiv x^\kappa \pmod{p},$$

because c is a polynomial; hence the restarted recurrence *reduces modulo p to the original recurrence*, and $(\tilde{u}_0, \tilde{u}_1) = (0, 1) = (A_0, A_1)$. Therefore

$$\tilde{u}_x \equiv A_x \pmod{p} \quad (0 \leq x \leq p-1), \quad (6)$$

A_x being p -integral for $x < p$.

Step 4 (the boundary step). The one remaining index is $x = p$, where the leading coefficient degenerates. Write the exact recurrence for $\tilde{u}$ at $m = (u+1)p - 1$:

$$((u+1)p)^\kappa \tilde{u}_p = c(m) \tilde{u}_{p-1} - \lambda m^\kappa \tilde{u}_{p-2}.$$

Since $m \equiv p-1 \pmod{p}$ we have $c(m) \equiv c(p-1)$, $m^\kappa \equiv (p-1)^\kappa$, while $\tilde{u}_{p-1} \equiv A_{p-1}$, $\tilde{u}_{p-2} \equiv A_{p-2}$ by (6). Hence, dividing the left side as $((u+1)p)^\kappa = (u+1)^\kappa p^\kappa$,

$$(u+1)^\kappa [p^\kappa \tilde{u}_p] \equiv c(p-1) A_{p-1} - \lambda (p-1)^\kappa A_{p-2} = p^\kappa A_p \pmod{p},$$

the last equality being the exact recurrence (1) at $n = p-1$ (where $(n+1)^\kappa = p^\kappa$). Thus $p^\kappa \tilde{u}_p \equiv (u+1)^{-\kappa} p^\kappa A_p \pmod{p}$.

Step 5 (insert the block-0 seed). By Proposition 1, $p^\kappa A_p \equiv B_{p-1}^{-1}$, so $p^\kappa \tilde{u}_p \equiv (u+1)^{-\kappa} B_{p-1}^{-1} \pmod{p}$.

Step 6 (compose). From (5), $p^\kappa G_u = W_{up}(p^\kappa \tilde{u}_p)$. Now $W_{up} = \lambda^{up}/(up+1)^\kappa$ with $(up+1)^\kappa \equiv 1$ and $\lambda^{up} \equiv \lambda^u$ (Fermat), whence

$$p^\kappa G_u \equiv \lambda^u \cdot (u+1)^{-\kappa} B_{p-1}^{-1} = \frac{\lambda^u}{(u+1)^\kappa B_{p-1}} \pmod{p}. \quad \square$$

Remark 1. The key structural point is that no step divides by B_u . The obstruction posed by a run — a zero of B inside the block — is dissolved by (6), “the shifted second solution is congruent to the original one modulo p ”. This is why Lemma 4 is unconditional (subject only to $p \nmid \lambda B_{p-1}$) even across blocks that contain a zero.

6 Plücker compression: where a second step is needed

The single-step Casoratian G_u of Lemma 4 lets one advance the seed from block u to block $u+1$ whenever the endpoint value B_u (or, below, B_{u+1}) is a p -unit. To cross a block boundary sitting at a zero of B one needs the two-step cross-Casoratian

$$G_u^{(2)} := A_{(u+2)p} B_{up} - A_{up} B_{(u+2)p} \quad (0 \leq u \leq p-3),$$

the range guaranteeing $(u+2)p \leq (p-1)p < p^2$, so that $v_p(A_{(u+2)p}) \geq -\kappa$ and $p^\kappa G_u^{(2)}$ is p -integral. The following Plücker (Cramer) identity shows that $G_u^{(2)}$ is determined by the single-step data of Lemma 4 *except* at a zero.

Proposition 2 (Plücker compression). *Write $v(x) := (A_x, B_x)^\top$ and $D(x, y) := \det[v(x), v(y)] = A_x B_y - A_y B_x$. For any three indices, the Cramer relation in the two-dimensional solution space reads*

$$D(x, y) v(z) - D(x, z) v(y) + D(y, z) v(x) = 0.$$

At $(x, y, z) = ((u+2)p, (u+1)p, up)$ one has $D(x, y) = G_{u+1}$, $D(x, z) = G_u^{(2)}$ and $D(y, z) = G_u$, so the B -component reads, exactly,

$$G_{u+1} B_{up} - G_u^{(2)} B_{(u+1)p} + G_u B_{(u+2)p} = 0.$$

Multiplying by p^κ and reducing $B_{(u+i)p} \equiv B_{u+i} \pmod{p}$ (2) gives, for $0 \leq u \leq p-3$,

$$[p^\kappa G_u^{(2)}] B_{u+1} \equiv [p^\kappa G_{u+1}] B_u + [p^\kappa G_u] B_{u+2} \pmod{p}, \quad (7)$$

with right-hand side known from Lemma 4. Consequently, whenever $p \nmid B_{u+1}$, the value $p^\kappa G_u^{(2)}$ is determined by Lemma 4; the two-step seed is genuinely required only at the boundaries with $p \mid B_{u+1}$, that is, at a block whose left endpoint is a zero of B .

Proof. The Cramer relation holds because three vectors in a 2-dimensional space are linearly dependent with the stated determinant coefficients. The three determinants are the displayed cross-Casoratians by definition. All three products $[p^\kappa D(\cdot, \cdot)]B_{(\cdot)p}$ are p -integral ($v_p(D) \geq -\kappa$ since the indices are $< p^2$, and the B 's are integers), so the exact identity may be multiplied by p^κ and read modulo p , factor by factor, with (2) applied to the integer factors. If B_{u+1} is a p -unit, (7) solves for $p^\kappa G_u^{(2)}$. $\square$

Remark 2. Taking the same three points in the A -component (multiplied by $p^{2\kappa}$) gives an alternative derivation of the chain, but at a zero the coefficient of the two-step term does not vanish and the relation becomes circular; this is a second confirmation that the two-step seed is genuinely “one level deeper” precisely at a zero of B .

7 Hypothesis V6 and the induction chain

The chain that lifts Proposition 1 to all j runs on the single-step relation of Lemma 4. Put

$$X_j := p^\kappa A_{jp}, \quad Y_j := B_{p-1}^{-1} A_j.$$

Proposition 3 (induction chain). *Assume $p \nmid \lambda B_{p-1}$ and let $1 \leq j < p$. Write $r_{\min}(p) := \min Z(p)$, with $r_{\min}(p) := p - 1$ if $Z(p) = \emptyset$.*

(a) *Lemma 4 is equivalent to the one-step relation*

$$X_{j+1}B_j - X_jB_{j+1} \equiv B_{p-1}^{-1} \frac{\lambda^j}{(j+1)^\kappa} \pmod{p}, \quad (8)$$

and Y satisfies the identical relation exactly (its Casoratian is $W_j = \lambda^j/(j+1)^\kappa$).

(b) *If $p \nmid B_j$, then X_{j+1} is determined by (8) and $X_{j+1} \equiv Y_{j+1} \pmod{p}$ follows from $X_j \equiv Y_j$.*

(c) If $p \mid B_j$ (so $j \in Z(p)$) and B_{j-1} is a p -unit by Lemma 3), then X_{j+1} is reached from X_{j-1} by the two-step congruence

$$p^\kappa (A_{(j+1)p} B_{(j-1)p} - A_{(j-1)p} B_{(j+1)p}) \equiv B_{p-1}^{-1} \frac{c(j) \lambda^{j-1}}{j^\kappa (j+1)^\kappa} \pmod{p}, \quad (9)$$

which is precisely Hypothesis 1 (V6) below at $u = j - 1$ (its condition $p \mid B_{u+1}$ holds); the companion identity on the Y -side, $A_{j+1} B_{j-1} - A_{j-1} B_{j+1} = c(j) W_{j-1} / (j+1)^\kappa$, holds exactly.

With base cases $X_0 = 0 = Y_0$ and $X_1 \equiv Y_1$ (Proposition 1), the congruence $X_j \equiv Y_j \pmod{p}$, i.e.

$$p^\kappa A_{jp} \equiv B_{p-1}^{-1} A_j \pmod{p}, \quad (10)$$

holds unconditionally for every $1 \leq j \leq r_{\min}(p)$ — hence for all $1 \leq j < p$ when $Z(p) = \emptyset$ — and for every $1 \leq j < p$ under Hypothesis 1.

Proof. (a) Expand $G_j = A_{(j+1)p} B_{jp} - A_{jp} B_{(j+1)p}$, multiply by p^κ , and reduce $B_{jp} \equiv B_j$, $B_{(j+1)p} \equiv B_{j+1}$ (2); Lemma 4 becomes exactly (8). The exact Casoratian $A_{j+1} B_j - A_j B_{j+1} = W_j$ multiplied by B_{p-1}^{-1} shows Y satisfies (8). (b) If B_j is a unit, (8) for X and its Y -analogue, subtracted, give $(X_{j+1} - Y_{j+1}) B_j \equiv (X_j - Y_j) B_{j+1} \pmod{p}$, so $X_j \equiv Y_j \Rightarrow X_{j+1} \equiv Y_{j+1}$. (c) The left side of (9) is $p^\kappa G_{j-1}^{(2)}$ and the right side is the value asserted by Hypothesis 1 at $u = j - 1$; the Y -companion follows by expanding A_{j+1} , B_{j+1} once through (1) and using $A_j B_{j-1} - A_{j-1} B_j = W_{j-1}$. As B_{j-1} is a unit, subtracting $B_{p-1}^{-1} \times (Y\text{-companion})$ from (9) gives $(X_{j+1} - Y_{j+1}) B_{j-1} \equiv (X_{j-1} - Y_{j-1}) B_{j+1} \pmod{p}$, which transports $X_{j-1} \equiv Y_{j-1}$ across the zero.

For the aggregate claim, run the chain upward from the base cases. Every transition $j \mapsto j+1$ with $j < r_{\min}$ is of type (b) (no zero has yet been met), so $X_j \equiv Y_j$ holds unconditionally for all $j \leq r_{\min}$; note that $X_{r_{\min}}$ itself is still reached by a type (b) step, since $B_{r_{\min}-1}$ is a unit. The first transition of type (c) occurs at $j = r_{\min} \in Z(p)$ and invokes (9); beyond the first zero every further value depends on that invocation, so the unconditional range is exactly $j \leq r_{\min}$, and under Hypothesis 1 the chain continues through every zero to all $j < p$. $\square$

By Proposition 2, the value $p^\kappa G_u^{(2)}$ is a consequence of Lemma 4 whenever $p \nmid B_{u+1}$. At the transitions of type (c) one has $u+1 = j \in Z(p)$ — exactly the excluded case. The required input is isolated as follows.

Hypothesis 1 (V6). For every sporadic family and every prime $p \geq 5$ with $p \nmid \lambda B_{p-1}$: for every u with $p \mid B_{u+1}$ (necessarily $0 \leq u \leq p-3$, by Lemma 3), the two-step congruence

$$p^\kappa G_u^{(2)} \equiv B_{p-1}^{-1} \frac{c(u+1) \lambda^u}{(u+1)^\kappa (u+2)^\kappa} \pmod{p}$$

holds.

Hypothesis 1 has been verified exhaustively (693 cases, zero failures; Section 11). It is *not* a formal consequence of Lemma 4: a backward-mirror analysis of $G_u^{(2)}$ at a zero determines, modulo p , that the backward $(1,0)$ -solution dies across the boundary and that the coefficient of the surviving branch is B_{p-1}^{-1} , and yields the boundary relation $p^\kappa w_{p+1} \equiv 0$; but the residue $w_{p+1} \bmod p$ itself is *not* determined by mod- p information alone. Concretely, a forward shift identifies $w_{p+1} \equiv \lambda^{-1}((u+2)/(u+1))^\kappa \bar{u}_{p+1}$ with $\bar{u}_x \equiv B_{x-1} \pmod{p}$ ($x \leq p$), and $p^\kappa \bar{u}_{p+1}$ reduces to a shifted mod- p^2 Gessel congruence. Thus a proof of Hypothesis 1 requires a mod- p^2 (indeed mod- p^3) input of Gessel–Straub type: Straub’s expansion [12] $A(pn+k) \equiv A(k)A(n) + pn A'(k)A(n) \pmod{p^2}$ applied in the shifted window, read against the boundary recurrence to one higher power of p . We record it as a hypothesis and do not carry out this step here.

8 The main theorem

Theorem 1 (twisted Dwork seed). *Let $(A_n), (B_n)$ be the two solutions of a sporadic recurrence (1) and let $p \geq 5$ be a prime with $p \nmid \lambda B_{p-1}$. Then*

$$p^\kappa A_{jp} \equiv B_{p-1}^{-1} A_j \pmod{p}$$

holds unconditionally for every $1 \leq j \leq r_{\min}(p)$ — in particular for all $1 \leq j < p$ whenever $Z(p) = \emptyset$ — and for every $1 \leq j < p$ under Hypothesis 1. Equivalently the seed constant of (3) is $C_0(p) \equiv B_{p-1}^{-1} \pmod{p}$, and $C_1(p) \equiv b C_0(p)$.

Proof. The seed (10), with the stated unconditional range, is Proposition 3: its only conditional steps are the type-(c) transitions at the zeros of B , and the input needed there is exactly Hypothesis 1 (Proposition 2 covers every other boundary). The relation $C_1 = b C_0$ is the boundary degeneration: multiplying (1) at $n = jp$ by p^κ and reducing mod p , the leading coefficient $(jp+1)^\kappa \equiv 1$, the middle coefficient $c(jp) \equiv c(0) = b$, and the trailing term carries the factor $(jp)^\kappa$ of valuation κ and dies; hence $p^\kappa A_{jp+1} \equiv b p^\kappa A_{jp}$, i.e. $C_1 = b C_0$ (the same identity $B_1 = c(0) = b$ holds on the integer side). $\square$

Corollary 1 (unconditional primes). *If $Z(p) = \emptyset$ (no zero of B in $[0, p)$), then $p^\kappa A_{jp} \equiv B_{p-1}^{-1} A_j \pmod{p}$ for all $1 \leq j < p$ unconditionally. Under a Malik–Straub-type heuristic [6] the density of such primes is $e^{-1/2}$.*

Proof. If $Z(p) = \emptyset$ then $r_{\min}(p) = p-1$ and every transition in Proposition 3 is of type (b); Hypothesis 1 is never invoked. $\square$

Corollary 2 (comparison of the two seed routes). *For the two families with trivial character ($\mathbf{D}$ = the $\zeta(2)$ Apéry sequence A005258, and Apéry’s $\zeta(3)$ sequence A005259, where $\kappa = 3$, $\lambda = 1$) and every prime $p \geq 5$ with $p \nmid B_{p-1}$: $B_{p-1} \equiv 1 \pmod{p}$.*

Proof. For these families the Ljunggren-route seed of the companion work [14, Prop. S_0, T_0] gives, unconditionally, $p^\kappa A_p \equiv A_1 = 1 \pmod{p}$ (with A the second solution normalised by $A_1 = 1$). Comparing with the $j = 1$ Casoratian computation $p^\kappa A_p \equiv B_{p-1}^{-1}$ of Proposition 1 gives $B_{p-1}^{-1} \equiv 1$, i.e. $B_{p-1} \equiv 1 \pmod{p}$. $\square$

Remark 3. Corollary 2 assumes the nonvanishing of B_{p-1} that it then pins to the value 1. The nonvanishing itself (indeed $B_{p-1} \equiv 1$ outright, with no hypothesis) is proved by a direct Lucas-theorem computation in Proposition 4(i)–(ii) below; the corollary then serves as a consistency check between the two independent seed computations. Classically the two congruences are known in the much stronger form $B_{p-1} \equiv 1 \pmod{p^3}$ (Chowla–Cowles–Gessel, Beukers; see Section 9).

9 Character identification: $B_{p-1} \equiv \chi(p)$

Theorem 1 reduces the character twist (3) to a single residue of the *integer* solution:

$$C_0(p) \equiv B_{p-1}^{-1} \pmod{p}, \quad \text{hence} \quad C_0(p) = \chi(p) \iff B_{p-1} \equiv \chi(p) \pmod{p}. \quad (11)$$

(The last equivalence uses $\chi(p) = \pm 1$, so $\chi(p)^{-1} = \chi(p)$.) Thus the observed classification (3) is exactly the statement

$$B_{p-1} \equiv \chi(p) \pmod{p} \quad \text{in each sporadic family.} \quad (12)$$

We now prove (12) for five of the six families. The engine is the elementary congruence $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$ ($0 \leq k \leq p-1$), applied to the classical binomial representations of the integer solutions (see Zagier [15]; each representation below was also verified against the recurrence for all $n \leq 40$).

Proposition 4 (full-index character congruences). *For every prime $p \geq 5$, the congruence (12) holds in the families **A**, **B**, **C**, **D**, **E**; the same computation gives $B_{p-1} \equiv 1 \pmod{p}$ for Apéry’s $\zeta(3)$ sequence A005259. In particular the hypothesis $p \nmid B_{p-1}$ of Theorem 1 holds automatically in these five families.*

Proof. (**A**) $B_n = \sum_k \binom{n}{k}^3$, so $B_{p-1} \equiv \sum_{k=0}^{p-1} (-1)^{3k} = \sum_{k=0}^{p-1} (-1)^k = 1$.

(**D**) $B_n = \sum_k \binom{n}{k}^2 \binom{n+k}{k}$. For $1 \leq k \leq p-1$ the integer $p-1+k = p+(k-1)$ has base- p digits $(1, k-1)$ while k has digits $(0, k)$, so Lucas’ theorem gives $\binom{p-1+k}{k} \equiv \binom{1}{0} \binom{k-1}{k} = 0 \pmod{p}$. Only $k=0$ survives: $B_{p-1} \equiv 1$. The $\zeta(3)$ sequence $\sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$ is identical.

(**C**) $B_n = \sum_k \binom{n}{k}^2 \binom{2k}{k}$, so $B_{p-1} \equiv \sum_{k=0}^{p-1} \binom{2k}{k} \pmod{p}$, and $\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \chi_{-3}(p) \pmod{p^2}$ is the congruence of Sun–Tauraso [13].

(**B**) $B_n = \sum_{3k \leq n} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}$. At $n = p-1$: $\binom{p-1}{3k} \equiv (-1)^{3k} = (-1)^k$ cancels the sign, and $3^{p-1-3k} \equiv 27^{-k}$ by Fermat, so $B_{p-1} \equiv \sum_{0 \leq 3k \leq p-1} \frac{(3k)!}{k!^3 27^k}$. The range may be extended to $k \leq p-1$: for $3k \geq p$ (with $k < p$) the base- p addition $k+k+k$ carries, so $p \mid (3k)!/k!^3$ by Kummer’s theorem. Hence $B_{p-1} \equiv \sum_{k=0}^{p-1} \frac{(3k)!}{k!^3 27^k} \equiv \chi_{-3}(p) \pmod{p}$, the last congruence being the Rodriguez-Villegas conjecture for the truncated ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1)$ proved by Mortenson [7, 8].

(**E**) $B_n = \sum_{2k \leq n} \binom{n}{2k} \binom{2k}{k}^2 4^{n-2k}$. At $n = p-1$: $\binom{p-1}{2k} \equiv (-1)^{2k} = 1$ and $4^{p-1-2k} \equiv 16^{-k}$, so $B_{p-1} \equiv \sum_{0 \leq 2k \leq p-1} \binom{2k}{k}^2 16^{-k}$; the range extends to $k \leq p-1$ since $p \mid \binom{2k}{k}$ for $(p-1)/2 < k < p$, and $\sum_{k=0}^{p-1} \binom{2k}{k}^2 16^{-k} \equiv \chi_{-4}(p) \pmod{p}$ is again a Rodriguez-Villegas congruence proved by Mortenson [7, 8].

All five congruences are verified directly for every $p \leq 200$, as are the binomial representations and each intermediate reduction above (script `test_char_proofs.py`). $\square$

For the families **A** and **D** much more is classical: the congruence $B_{p-1} \equiv 1 \pmod{p^3}$ for A005258 and A005259 is due to Chowla, Cowles and Gessel (c. 1979–82) and Beukers [1], both stated explicitly in Beukers’ survey [2, §3]; for the Franel numbers, congruences of this type go back to Gessel [4] and Coster [3] (catalogued in Osburn–Sahu–Straub [10]). Proposition 4 only claims the mod- p statements, which is all the seed theory needs.

*The remaining family **F**.* For the half-index value $B_{(p-1)/2}$ deep results are known — the Stienstra–Beukers congruence [11] in family **D**, the Beukers–Ahlgren–Ono line for A005259, and, for family **F** itself, the congruence $B_{(p-1)/2} \equiv \gamma(p) \pmod{p}$ proved by Kazalicki [5] — but these concern

the half-index, not (12). For the full-index form $B_{p-1} \equiv \chi_{-3}(p) \pmod{p}$ in family **F** we were unable to locate an explicit statement in the literature, nor a binomial representation to which the Lucas reduction above applies; it is verified numerically for all $p \leq 200$ and appears to be new. (Osburn–Sahu [9, Remark 3.1] relate the generating series of family **C** to a weight-3 Eisenstein series with character χ_{-3} ; a similar Eisenstein-series mechanism, as used by Kazalicki in the half-index setting, is the natural candidate route for **F**.)

Remark 4 (a naming caveat). The sequence A002893 (family **C**) is occasionally called “Domb”, but the standard *Domb numbers* are A002895 = $\sum_k \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}$, which are *not* among Zagier’s six sporadic sequences (they satisfy a different, third-order relation). The congruence $D_{p-1} \equiv 1 \pmod{p^3}$ of Z.-H. Sun (partly proved by Mao–Schlosser) concerns A002895, not A002893; the two must not be conflated.

Remark 5 (archimedean counterpart). Zagier [15, §3] describes the generating functions of the second solutions as Eichler integrals of weight-3 Eisenstein series twisted by precisely the characters of (3), the corresponding special values being rational multiples of $\zeta(2)$ (**A**, **D**), $L(\chi_{-3}, 2)$ (**B**, **C**, **F**) and $L(\chi_{-4}, 2)$ (**E**). Identity (11) exhibits the same character at the finite places as the value $\chi(p)$: the seed congruence is a p -adic counterpart of Zagier’s Eichler-integral description of the second solution.

10 Application to the gcd structure theorem

The seed congruence (10) combines with a propagation principle to yield the prime structure of $\gcd(d_n A_n, d_n B_n)$, $d_n = \text{lcm}(1, \dots, n)^\kappa$, in each sporadic family. The propagation principle is that of the companion work; we state it for completeness.

Proposition 5 (propagation). *Let $\sqrt{n} < p \leq n$, $n = jp + n_0$, $p \nmid \lambda B_{p-1}$. Set $X(n_0) := p^\kappa A_{jp+n_0}$. Then, granting the seed (10),*

$$X(n_0) \equiv B_{n_0} C_0(p) A_j \pmod{p} \quad (0 \leq n_0 \leq p-1), \quad C_0(p) \equiv B_{p-1}^{-1}.$$

Proof. Multiplying (1) at index $m = jp + n_0$ by p^κ and reducing mod p gives, for $1 \leq n_0 \leq p-2$,

$$(n_0 + 1)^\kappa X(n_0 + 1) \equiv c(n_0) X(n_0) - \lambda n_0^\kappa X(n_0 - 1) \pmod{p},$$

since the coefficients of (1) are polynomials in the index and all $p^\kappa A_{jp+n_0}$ are p -integral ($jp + n_0 \leq p^2 - 1$). The leading coefficient is invertible for

$n_0 \leq p-2$, so $X(n_0+1)$ is determined by its two predecessors. The sequence $Y(n_0) := B_{n_0}X(0)$ satisfies the same recurrence (the integer recurrence for B times the constant $X(0)$). The seeds match: $X(0) = Y(0)$ trivially, and $X(1) \equiv bX(0) = B_1X(0) = Y(1)$ by the boundary degeneration ($C_1 = bC_0$, Theorem 1). Induction gives $X(n_0) \equiv B_{n_0}X(0)$, and $X(0) = p^\kappa A_{jp} \equiv C_0(p)A_j$ by (10). $\square$

Theorem 2 (sporadic gcd structure). *Let $(A_n), (B_n)$ be the two solutions of a sporadic recurrence, $\kappa = 2$, $d_n = \text{lcm}(1, \dots, n)^2$, and $\sqrt{n} < p \leq n$ with $p \nmid \lambda$, $n = jp + n_0$. Assume $p \nmid B_{p-1}$. Then*

$$v_p(\gcd(d_n A_n, d_n B_n)) > 0 \iff p \mid B_{n_0} \text{ or } p \mid \text{num}(A_j),$$

unconditionally for every prime with $Z(p) = \emptyset$, and in general under Hypothesis 1.

Proof. As in [14], $v_p(d_n) = \kappa$ and $v_p(B_n) \geq 0 > -\kappa$, so $v_p(\gcd) = \kappa + \min(v_p(A_n), v_p(B_n)) > 0 \iff v_p(A_n) > -\kappa \iff p \mid X(n_0)$. By Proposition 5, $X(n_0) \equiv B_{n_0}C_0(p)A_j$ with $C_0(p) \equiv B_{p-1}^{-1}$ a p -unit; since A_j is p -integral ($j < p$), $p \mid X(n_0) \iff p \mid B_{n_0}$ or $p \mid \text{num}(A_j)$. The character twist $C_0(p) = \chi(p)$, being a unit, does not affect the criterion. The conditional/unconditional dichotomy is that of Theorem 1 and Corollary 1. Small primes $p \leq \sqrt{n}$ contribute $O(\sqrt{n} \log n)$ as in the $\zeta(3)$ case; the family **D** instance is Theorem Z2 of [14], and the $\zeta(3)$ analogue is proved there unconditionally. $\square$

Two refinements are worth recording. First, for a given pair (n, p) the criterion only requires the seed at the single value $j = \lfloor n/p \rfloor$; by Theorem 1 it therefore holds unconditionally whenever $\lfloor n/p \rfloor \leq r_{\min}(p)$, even if $Z(p) \neq \emptyset$. Second, by Proposition 4 the hypothesis $p \nmid B_{p-1}$ is a theorem in the families **A**, **B**, **C**, **D**, **E**, so that in those five families the statement of Theorem 2 carries no hypothesis beyond Hypothesis 1 (and none at all at primes with $Z(p) = \emptyset$ or when $\lfloor n/p \rfloor \leq r_{\min}(p)$).

This upgrades the sporadic conjecture of [14, Conj. 1] to a theorem for every prime with $Z(p) = \emptyset$, and to a theorem modulo the single Hypothesis 1 in general. The $\zeta(2)$ and $\zeta(3)$ cases (family **D** and Apéry's own sequence) are proved unconditionally in the companion work by a Ljunggren-based seed, to which we refer for the explicit-formula route.

11 Numerical verification

All computations are in exact arithmetic (Python/sympy); scripts are `test_casoratian_route.py` (V1–V4) and `test_cross_casoratian.py` (V5–V6). Across the two scripts, 2380 prime-index checks were performed with *zero* failures.

- **V1 (Lemma 1).** The Casoratian formula (4) and $W_n = \lambda^n/(n+1)^\kappa$: 21 checks, 0 failures.
- **V2 (identity (12)).** $B_{p-1} \equiv \chi(p) \pmod{p}$ across all six families and the stated characters: 105 checks, 0 failures.
- **V3 (seed (10)).** $p^\kappa A_{jp} \equiv B_{p-1}^{-1} A_j \pmod{p}$: 833 checks, 0 failures.
- **V4 (zero terrain).** Existence and non-adjacency of the zeros $Z(p)$ (Lemmas 2, 3), including deep zeros $v_p(B_r) \geq 2$ (e.g. family **D** at $r = (p-1)/2$ for every $p \equiv 3 \pmod{4}$, the mod- p^2 shadow of the Stienstra–Beukers structural zero); no adjacent zeros were found.
- **V5 (Lemma 4).** $p^\kappa G_u \equiv \lambda^u/((u+1)^\kappa B_{p-1}) \pmod{p}$: 728 checks (including 44 across blocks containing a run), 0 failures.
- **V6 (Hypothesis 1).** The two-step cross-Casoratian at zeros: 693 checks, 0 failures. This is verified but *not* proved; see Section 7.

Status summary. Proved (for $p \geq 5$ and, where stated, $p \nmid B_{p-1}$ — itself a theorem in families **A–E** by Proposition 4): Lemmas 1, 2, 3, Propositions 1, 2, 3, 5, 4, Lemma 4, Theorem 1 and Theorem 2 in their unconditional ranges ($j \leq r_{\min}(p)$; all j when $Z(p) = \emptyset$), and Corollaries 1, 2. Verified numerically but not proved: Hypothesis 1 (693 cases), and the identity (12) for family **F** ($p \leq 200$). The gcd structure theorem (Theorem 2) is therefore unconditional exactly at primes with $Z(p) = \emptyset$ (or when $\lfloor n/p \rfloor \leq r_{\min}(p)$) and conditional on Hypothesis 1 otherwise.

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